

Chromatic and Structural Properties of Sparse Graph Classes

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work, with the following exceptions:

- Chapter 2 is joint work with Jan van den Heuvel, Patrice Ossona de Mendez, Roman Rabinovich, and Sebastian Siebertz. It is based on published work [39].
- Chapter 3 is joint work with Jan van den Heuvel and Hal Kierstead. It is based on work accepted for publication [37].
- Chapter 5 is joint work with Jan van den Heuvel, Stephan Kreutzer, Michał Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. It is based on published work [38].

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Abstract

A *graph* is a mathematical structure consisting of a set of objects, which we call vertices, and links between pairs of objects, which we call edges. Graphs are used to model many problems arising in areas such as physics, sociology, and computer science.

It is partially because of the simplicity of the definition of a graph that the concept can be so widely used. Nevertheless, when applied to a particular task, it is not always necessary to study graphs in all their generality, and it can be convenient to studying them from a restricted point of view. Restriction can come from requiring graphs to be embeddable in a particular surface, to admit certain types of decompositions, or by forbidding some substructure. A collection of graphs satisfying a fixed restriction forms a *class* of graphs.

Many important classes of graphs satisfy that graphs belonging to it cannot have many edges in comparison with the number of vertices. Such is the case of classes with an upper bound on the maximum degree, and of classes excluding a fixed minor. Recently, the notion of *classes with bounded expansion* was introduced by Nešetřil and Ossona de Mendez [62], as a generalisation of many important types of sparse classes. In this thesis we study chromatic and structural properties of classes with bounded expansion.

We say a graph is k -degenerate if each of its subgraphs has a vertex of degree at most k . The degeneracy is thus a measure of the density of a graph. This notion has been generalised with the introduction, by Kierstead and Yang [47], of the *generalised colouring numbers*. These parameters have found applications in many areas of Graph Theory, including a characterisation of classes with bounded expansion. One of the main results of this thesis is a series of upper bounds on the generalised colouring numbers, for different sparse classes of graphs, such as classes excluding a fixed complete minor, classes with bounded genus and classes with bounded tree-width.

We also study the following problem: for a fixed positive integer p , how many colours do we need to colour a given graph in such a way that vertices at *distance exactly* p get different colours? When considering classes with bounded expansion, we improve dramatically on the previously known upper bounds for the number of colours needed.

Finally, we introduce a notion of *addition of graph classes*, and show various cases in which sparse classes can be summed so as to obtain another sparse class.

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“I would maintain that thanks are the highest form of thought, and that gratitude is happiness doubled by wonder.” – Gilbert K. Chesterton

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Chapter 1

Introduction

In the same year as the Great Fire of London, a young Leibniz presented, as part of his doctoral thesis, his *Dissertatio de arte combinatoria* [58]. While the present thesis shares with that of Leibniz that it deals with problems which fall within the broad area of Combinatorics, we cannot claim, as Leibniz did in the front page of his dissertatio, that we “prove the existence of God with complete mathematical certainty”. Here, we focus strictly on a branch of Combinatorics known as Graph Theory.

A graph is an abstract structure consisting of a set of objects, which we call *vertices*, and a set of relationships between certain pairs of vertices, which we call *edges*. Graphs are used to model many “real life” problems related with telecommunications, transportation, chemistry, sociology, etc. But they are also used to model equally real, although (at least immediately) less practical, questions that people, when allowed to be at leisure, tend to contemplate.

Aristotle [7] attributes the “founding” of mathematics to the priestly caste of ancient Egypt, for this “caste was allowed to be at leisure”. Indeed, Francis Guthrie (1831-1899) must have enjoyed time of leisure when, being roughly 21 years old, he conjectured what is now the Four Colour Theorem, while colouring the map of England. That is, he guessed that four colours suffice if one wants to colour any map which can be drawn on a plane, in such a way that neighbouring countries get different colours. If we model countries as vertices and add an edge between each pair of vertices corresponding to neighbouring countries, we can see this problem as a *vertex colouring problem*. This problem, which remained unsolved for over a century, is the non plus ultra example of a problem in Chromatic Graph Theory. This

branch of Graph Theory can be said to be concerned with the conditions in which a certain set of labels or “colours” can be assigned to a graph. We shall study various problems of this type.

The other kind of problems we will address here, belong to what is known as Structural Graph Theory. Problems in this area are concerned with the description or characterisation of graphs having certain restrictions on the way edges between pairs of vertices can be present. Well-known examples of this are the characterisations of graphs which come from modelling maps on a plane. Kuratowski’s and Wagner’s theorems characterise planar graphs in term of forbidden substructures. Structural problems enjoy a privileged position within Graph Theory as they are closely related to Theoretical Computer Science. Problems which cannot be solved “efficiently” in general, may be solved efficiently in graphs of a given class, if these graphs have certain structure: they have restrictions on the edge density, exclude some substructure, or allow some decomposition.

Chromatic and structural properties of graphs are often very closely linked, and their interplay will be at the heart of our thesis. We shall see how more restricted graph classes allow a more thorough study of chromatic properties. And, most importantly, we shall also see how considering the right structural property can allow us to prove very surprising results for many important graph classes. All of this within a framework of the study of graph classes which has seen great developments in the last 90 years.

This first, introductory, chapter will be organised as follows. In Section 1.1, we give an introduction to various relations and parameters which are frequently used to define graph classes. By mentioning some important results, we aim to convince the reader of the importance of studying the graph classes which form the framework of this thesis. Some of these results are related to the efficiency of algorithms in certain graph classes, and particularly motivate our results in Chapter 5. In Section 1.2, we study the relationship between chromatic number, degeneracy, and edge density, using this as motivation for the introduction of the generalised colouring numbers. We also survey a few results which relate the generalised colouring numbers to various important graph classes, mentioning the contributions we make in Chapter 2. In Section 1.3, we introduce graph powers, exact powers, and exact distance graphs, alluding to some results and constructions which serve as context for our results in

Chapters 3 and 4. We also make explicit some of the results contained in these two chapters. Finally, in Section 1.4, we give summaries of the chapters of the thesis.

1.1 Subgraphs, minors and classes with bounded expansion

We assume some familiarity with basic notions and standard notation of Graph Theory. For this we refer the reader to Chapter 1 of [18], which is available online. Nevertheless, for the sake of completeness, we shall introduce some basic definitions and notation, especially in the present chapter.

As mentioned earlier, a graph $G = (V(G), E(G))$, is an abstract structure which consists of a set of vertices $V(G)$, and a set of edges $E(G)$ formed by pairs of distinct elements in $V(G)$. If the graph under consideration is clear, we may denote these sets by V and E , respectively. All the graphs considered in this thesis will be finite, undirected and will have no loops or multiple edges.

Let G be a graph and let $u, v \in V(G)$. If the pair $uv = e$ is an edge in $E(G)$, we say that u and v are *adjacent* or, equivalently, that they are *neighbours*. We also say that u and v are the *endpoints* of the edge e . The *neighbourhood* $N(w)$ of a vertex $w \in V(G)$ is the set of vertices adjacent to w in G , and the *degree* of w is the size of $N(w)$. If a vertex w is an endpoint of an edge e' , we say that e' is *incident* to w . A *walk* in G is a sequence of vertices v_1, v_2, \dots, v_k such that $v_i v_{i+1} \in E(G)$ for all $1 \leq i \leq k-1$. We say that v_1 and v_k are the *endpoints* of the walk; while v_2, \dots, v_{k-1} are the *internal vertices* of the walk. If the walk satisfies $v_i \neq v_j$ for all values $1 \leq i < j \leq k$, then we say it is a *path*. If we have $v_i = v_j$ for $(i, j) = (1, k)$, and for no other pair of indices, then the walk is called a *cycle*. The *length* of a walk v_1, v_2, \dots, v_k is the number of edges it traverses (i.e. $k-1$). We call a graph G *connected* if, for every pair of vertices $u, v \in V(G)$, there is a path P_{uv} in G such that u and v are the endpoints of P_{uv} .

The *complete graph* (also *clique*) K_n is the graph such that $|V(K_n)| = n$, and $uv \in E(G)$ for all pairs of vertices $u, v \in V(G)$. The *complete bipartite graph* $K_{m,n}$ is the graph having a partition of $V(G)$ into two sets A and B , such that $|A| = m$, $|B| = n$, $uv \in E(G)$ if $u \in A$ and $v \in B$, and $wz \notin E(G)$ if w and z belong to the same part. A graph T is said to be a *tree* if it is connected and $|E(T)| = |V(T)| - 1$. A *rooted tree* is a tree T in which a vertex

$v \in V(T)$ has been labelled as the *root*.

For a positive integer k , we denote $[k] = \{1, 2, \dots, k\}$.

For a graph $G = (V, E)$ we define the following operations:

- By *removing a vertex v* we mean removing v from V and removing from E every edge incident to v .
- By *removing an edge e* we mean removing e from E .
- By *subdividing an edge e* we mean removing e from E and adding, between the endpoints of e , a path with new edges and vertices, except for the endpoints which are the same endpoints of e . We say we have *subdivided an edge k times*, if the corresponding path has length $k + 1$.

From these operations we can define three different “containment” relationships between graphs.

- A graph H is an *induced subgraph* of a graph G , if H can be obtained from G by a series of vertex deletions. The set of vertices $W \subseteq V(G)$ which was not removed is said to *induce H* , and we denote this by $G[W] = H$.
- A graph H is a *subgraph* of a graph G , if H can be obtained from G by a series of vertex deletions and edge deletions. If a graph A contains B as a subgraph, then A is a *supergraph* of B .
- We say a graph A is a *subdivision* of a graph B if it can be obtained from B by a series of subdivisions. A graph H is a *topological minor* of a graph G if G contains a subdivision of H as a subgraph.

We clearly have that if H is an induced subgraph of G , it is also a subgraph of G ; and if H is a subgraph of G , then it is also a topological minor of G . We now define the notion of *minor* which (*immersions* aside) completes the list of the most well known containment relations for graphs.

Let $G = (V, E)$ be a graph and $\mathcal{H} = \{H_1, \dots, H_m\}$ a collection of non-empty connected subgraphs of G such that the vertex sets $V(H_1), \dots, V(H_m) \subseteq V$ are disjoint. We denote

by G/\mathcal{H} the graph with vertex set $\{1, 2, \dots, m\}$ having ij as an edge if and only if there is an edge in E between an vertex of $V(H_i)$ and a vertex of $V(H_j)$. A graph H is a *minor* of G if H is a subgraph of G/\mathcal{H} for some \mathcal{H} . In this case, we say that the subgraphs H_1, \dots, H_m of \mathcal{H} form a *minor model* of H in G . If there is no \mathcal{H} such that $H = G/\mathcal{H}$, then we say G is H -minor free.

Informally, a graph is said to be *planar* if it can be drawn in the plane in such a way that no edges intersect, except at a common endpoint. Two very famous results characterise planar graphs in terms of graph they cannot “contain”.

Theorem 1.1.1 (Kuratowski [56]).

A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a topological minor.

Theorem 1.1.2 (Wagner [81]).

A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.

The fact that these two theorems coexist is surprising. If a graph G contains a graph H as a topological minor, then G contains H as a minor, but the reverse is false even for $H = K_5$. Therefore, the class of graphs excluding H as a minor is usually more restricted than the class of graphs excluding H as a topological minor.

Another important aspect of these theorems is that the characterisation comes through a *finite* set of forbidden substructures. We say a property is *closed* under a containment relationship (say, subgraphs) if for every graph G that has the property, all the graphs that G contains, under this relationship, also have the property. If the class of graphs having a certain property (such as planarity) is closed under a containment relationship and it can be characterised by a finite number of forbidden structures, then checking whether a graph belongs to this class is reduced to checking the containment of these substructures. A fact that sets the minor relationship apart from the other three containment relationships is that *every* minor closed property can be characterised in this way. This is in fact a corollary of the following result, which Diestel [18] says “may doubtless be counted among the deepest theorems mathematics has to offer”.

Theorem 1.1.3 (Robertson and Seymour [72]).

For every infinite sequence G_1, G_2, \dots of graphs, there are indices $i < j$ such that G_i is a minor of G_j .

Corollary 1.1.4.

Let P be a property of graphs that is closed under the minor relationship. Then there exists a finite collection of graphs H_1, \dots, H_k , such that for every graph G we have: G has property P if and only if G contains none of H_1, \dots, H_k as a minor.

The Robertson-Seymour theorem (previously known as Wagner’s conjecture) was the main goal of a monumental project which spanned over 20 papers, published in a period of over two decades. The tools and results developed in this project have supported and shaped much of the research done since and will undoubtedly continue to do so. This is the case not just within Graph Theory, but also beyond, particularly within Theoretical Computer Science. For instance, Robertson and Seymour proved [71] that, for a fixed graph H , there is a polynomial time algorithm for checking whether a given graph G contains H as a minor. This means, together with Corollary 1.1.4, that there is a polynomial time algorithm for checking if a graph belongs to a given minor closed class. However, the algorithm provided by Robertson and Seymour, as they themselves put it, “is not practically feasible, since it involves the manipulation of enormous constants” [71]. Moreover, the algorithm is not constructive, in the sense that it does not give a way of obtaining an excluded minor characterisation of a minor closed class. This had lead to an ongoing search for extracting constructive and practical algorithmic results from the theory developed by Robertson and Seymour (see, e.g., [2, 16, 42]).

Apart from its corollaries and the research paths it opened, the theory developed by Robertson and Seymour is of great importance because of the tools it drew attention to. Perhaps the most famous examples of this are the notions of *tree-decompositions* and *tree-width*. A *tree-decomposition* of a graph G is a pair $(T, (X_y)_{y \in V(T)})$, where T is a tree and $X_y \subseteq V(G)$ for each $y \in V(T)$, such that

$$(1) \bigcup_{y \in V(T)} X_y = V(G);$$

$$(2) \text{ for every edge } uv \in E(G), \text{ there is a } y \in V(T) \text{ such that } u, v \in X_y; \text{ and}$$

- (3) if $v \in X_y \cap X_{y'}$ for some $y, y' \in V(T)$, then $v \in X_{y''}$ for all y'' that lie on the unique path between y and y' in T .

So as to avoid confusion with the vertices of G , we say that the elements of $V(T)$ are the *nodes* of T . For a node y we say that corresponding set X_y is the *bag* of y . The width of a tree-decomposition is $\max_{y \in V(T)} |X_y| - 1$, and the *tree-width* of G is equal to the smallest width of any tree-decomposition of G .

The notion of tree-width was first introduced by Halin [36], who showed, among other things, that grid graphs can have arbitrarily large tree-width. Robertson and Seymour [70] reintroduced it, apparently unaware of Halin's work, and it has since been a fundamental tool in Graph Theory, particularly for graph algorithms, as we shall illustrate below.

The difficulty of deciding if a graph has a certain property, can be seen as the amount of resources (e.g., running time or storage space) that this decision requires, no matter what algorithm is used. This *complexity* is generally measured as a function of the size of the graphs under consideration. However, since certain algorithms require, apart from the graph itself, some other input, the complexity of decision may be measured not only in terms of the input size, but also in terms of some other *parameter* of the graph. We may refine, then, the study of the complexity of a decision problem, by considering how it behaves for each fixed value of a given parameter.

For many decision problems, choosing a parameter that reflects some structural property reveals important aspects of the complexity. In fact, a large class of graph properties can be decided in linear time when parametrised in terms of the tree-width. A theorem of Courcelle [15], which is an example of what is known as an algorithmic meta-theorem, states that every graph property definable in monadic second-order logic can be evaluated in linear time on any fixed class of graphs of bounded tree-width. Many results in this thesis, although not algorithmic, will be “parametrised” in terms of the tree-width.

If a property is closed under taking minors, then we say that the class of graphs having that property is *minor closed*. Notice that the class of all graphs is minor closed (in fact, it is closed under any containment relationship). If a class is minor closed and is not the class of all graphs, then we say that it is a *proper minor closed* class. However, since in this thesis we deal mostly with restricted classes of graphs, we will usually omit the word “proper”.

If a graph has tree-width at most t , its minors have tree-width at most t . This implies, for instance, that the class of trees, which is the class of graphs having tree-width at most 1, is minor closed. However, if we take only the class of trees with maximum degree at most k , for some $k \geq 2$, then this class is not minor closed. Hence, there are classes with an upper bound on the maximum degree, which we will call *degree bounded classes*, that are not minor closed. And clearly there are minor closed classes which are not degree bounded (consider the class of trees). However, both minor closed and degree bounded classes are closed under taking topological minors. In [62], Nešetřil and Ossona de Mendez introduced the notion of *graph classes with bounded expansion* which generalises classes closed under taking topological minors and, thus, many of the classes we have here presented. We now give a definition of these classes.

For a graph G and vertices $u, v \in V(G)$, let $d_G(u, v)$ denote the distance between u and v in G , that is, the number of edges contained in a shortest path between u and v . (We may write $d(u, v)$ instead of $d_G(u, v)$ if the graph under consideration is clear.) For a vertex $v \in V(G)$, we denote by $N_G^k(v)$ (or just $N^k(v)$) the k -th *neighbourhood* of v , that is, the set of vertices different from v with distance at most k from v . We also set $N^k[v] = N^k(v) \cup \{v\}$ and call this the *closed k -th neighbourhood* of v . As is standard, we write $N(v)$ for $N^1(v)$.

Assuming G is connected, we can define the *radius* of G as

$$\rho(G) = \min_{v \in V(G)} \max_{x \in V(G)} d(x, v),$$

where we set $d(u, u) = 0$ for any $u \in V(G)$.

Let $\mathcal{H} = \{H_1, H_2, \dots, H_m\}$ be a collection of non-empty connected subgraphs of G such that the vertex sets $V(H_1), \dots, V(H_m) \subseteq V$ are disjoint. We let

$$\rho(\mathcal{H}) = \max\{\rho(H_i) \mid i = 1, 2, \dots, m\}.$$

Let r be a non-negative integer. We say that H is a *depth- r minor* of a graph G , if there is a collection \mathcal{H} of non-empty vertex-disjoint connected subgraphs of $V(G)$, with $\rho(\mathcal{H}) \leq r$, such that H is a subgraph of G/\mathcal{H} . (Notice that the depth-0 minors of G are its subgraphs.) The

grad of G of rank r is defined as

$$\nabla_r(G) = \max \frac{|E(H)|}{|V(H)|}$$

where the maximum is taken over all depth- r minors H of G .

Definition 1.1.5.

A class of graphs \mathcal{K} has bounded expansion if there is a function $f: \mathbb{N} \rightarrow \mathbb{R}$ such that for any graph $G \in \mathcal{K}$ and any non-negative integer r , we have $\nabla_r(G) \leq f(r)$.

The notion of grad allows for a refined treatment of the *edge density* $E(H)/V(H)$ of minors H of a graph G . A class with bounded expansion has, for all the graphs in it, an upper bound on the number of edges of a depth- r minor which is linear in the number of vertices of such a minor. This provides a very robust framework in which we can talk about “sparse” graph classes. For minor closed classes we can take f to be a constant function, while for degree bounded classes we may take f to be an exponential function. Indeed, there are classes with bounded expansion which are neither minor closed nor degree bounded. For any integer $i \geq 2$, construct the graph A_i from the complete graph K_i , by subdividing each edge $i - 2$ times. The class of graphs $\{A_i \mid i \geq 2\}$ contains all cliques as topological minors, and yet it has bounded expansion with corresponding function $f(r) = \binom{r+1}{2}/(r+1)$.

As with classes with bounded tree-width, it has been shown that a large (although more restricted) class of graph properties can be decided in linear time on any fixed class of bounded expansion [22, 31]. Similarly, many interesting results which are not true for the class of all graphs can be proved to be true in the very general context of classes with bounded expansion.

Since their introduction, many seemingly unrelated characterizations of classes with bounded expansion have been found. This provides many tools with which one can prove results for this type of classes. In particular, for minor closed classes, this provides alternatives to some of the tools developed by Robertson and Seymour which, versatile and powerful as they are, usually involve huge constants. The present thesis gives, particularly in Chapters 2, 3 and 4, various structural and chromatic results in the broad context of classes with bounded expansion.

These results are complemented, with the study in Chapter 5, of a notion of *addition of*

graph classes. In that chapter, we study different ways in which we can add edges to each graph of a class with bounded expansion, so as to guarantee that each new graph gets a nice structural property, and that the resulting class has bounded expansion. That is, we study convenient ways of pushing up the edge density of the graphs in a class with bounded expansion, while still keeping the nice algorithmic properties we have so far sketched, and the chromatic and structural properties we will describe in detail throughout the thesis.

1.2 Chromatic number, degeneracy and the generalised colouring numbers

We have already mentioned colourings, but let us define them formally. For a graph $G = (V, E)$ a (proper) *vertex colouring* is a map $c: V \rightarrow S$ such that $c(u) \neq c(v)$ whenever u and v are adjacent. The elements of S are called *colours*, and what we are mainly concerned with is the size of S . We say that c is a *k-colouring* if it is a vertex colouring where $|S| = k$. The *chromatic number* $\chi(G)$ of a graph G is the least k for which there is a k -colouring of $V(G)$.

A notion which is related to the chromatic number is the *degeneracy* of a graph. A graph G is *k-degenerate* if every subgraph of G contains a vertex of degree at most k . The *degeneracy* $\deg(G)$ of G is the least k such that G is k -degenerate. It is well known that for every graph G we have $\chi(G) \leq \deg(G) + 1$. To check this for a graph G , create a linear ordering of $V(G)$ by removing at each step a vertex of degree at most $\deg(G)$, and then reversing the order (so the first removed vertex is last in the order). Then, each vertex v has at most $\deg(G)$ neighbours which come before v in the ordering. Applying the greedy algorithm on the resulting ordering, we obtain a colouring of the graph G which uses at most $\deg(G) + 1$ colours. Thus, for every graph G we have $\chi(G) \leq \deg(G) + 1$. Although this bound is far from being tight in many cases (such as in the case of complete bipartite graphs), it is often used to show that a specific class of graphs has bounded chromatic number.

The degeneracy of a graph G is also closely related to its edge density $\nabla_0(G)$, that is, the grad of rank 0 defined in the previous subsection. The following result is folklore. (Note that $2\nabla_0(G)$ is usually referred to as the maximum average degree.)

Proposition 1.2.1.

Let k be a positive integer. For any graph G we have:

- (a) If $\lfloor 2\nabla_0(G) \rfloor \leq k$, then G is k -degenerate.
- (b) If G is k -degenerate, then $\nabla_0(G) < k$.

Let us give an example of how these notions can come together, and the important place they have in Graph Theory.

In 1943, Hugo Hadwiger [34], made a far-reaching generalisation of, what was at the time, the Four Colour Conjecture. He conjectured that every K_t -minor free graph has a $(t - 1)$ -colouring. This conjecture is widely considered to be one of the most important open problems in graph theory. Indeed, in 1980, Bollobás, Catlin and Erdős [12] referred to it as “one of the deepest unsolved problems in graph theory”. The Hadwiger Conjecture is known to be true only for $t \leq 6$, and it is not even known whether there is a constant c such that every K_t -minor free graph has a ct -colouring. In general, $ct\sqrt{\log t}$ is the best upper bound known on the chromatic number of K_t -minor free graphs, for some constant c . This bound was obtained by means of the following result. Kostochka [48] and Thomason [79] independently proved that there is a constant c such that every K_t -minor free graph G satisfies $\nabla_0(G) \leq ct\sqrt{\log t}$. Proposition 1.2.1 (a) and the fact that $\chi(G) \leq \deg(G) + 1$ complete the proof of the upper bound bound on the chromatic number.

The importance of the inequality $\chi(G) \leq \deg(G) + 1$ and of the algorithm which proves it, lead to the introduction of the *colouring number* $\text{col}(G)$. For a graph G , this is the minimum integer k such that there is a linear ordering L of $V(G)$ such that every vertex y has at most $k - 1$ neighbours x with $x <_L y$. Since the colouring number is one more than the degeneracy of a graph, we have $\chi(G) \leq \text{col}(G)$.

Different generalisations of the colouring number can be found in the literature. Chen and Schelp [14] proved that the class of planar graphs has linear Ramsey number by also controlling, for all vertices v , the number of smaller vertices (with respect to a fixed linear ordering) that can be reached by a path of length two, whose middle vertex is larger than v . Various versions of their idea were applied by Kierstead and Trotter [45], Kierstead [43], and Zhu [83] to problems concerning the game chromatic number of graphs and gave rise to the 2-colouring number defined below. In their study of oriented game chromatic number of

graphs, Kierstead and Trotter [46] considered paths of length four with different configurations of “large” internal vertices, which later motivated the notions of 4-colouring number and weak 4-colouring number. Kierstead and Kostochka [44] applied 2-colouring numbers to a (non-game) packing problem.

All of these notions are encompassed in the concepts of the *strong k -colouring number* and the *weak k -colouring number* of a graph, both of which were first introduced in their full generality by Kierstead and Yang [47].

Let $G = (V, E)$ be a graph, L a linear ordering of V , and k a positive integer. We say that a vertex $x \in V$ is *strongly k -reachable* from $y \in V$ if $x <_L y$ and there exists an xy -path P of length at most k such that $y <_L z$ for all internal vertices z of P . Similarly, if all internal vertices z of P satisfy the less restrictive condition that $x <_L z$, then we say that x is *weakly k -reachable* from y . Let $S_k[G, L, y]$ be the set of vertices that are strongly k -reachable from y , and $W_k[G, L, y]$ the set of vertices that are weakly k -reachable from y . The *strong k -colouring number* $\text{scol}_k(G)$ and *weak k -colouring number* $\text{wcol}_k(G)$ of a graph G are defined as follows:

$$\text{scol}_k(G) = 1 + \min_L \max_{y \in V} |S_k[G, L, y]|,$$

$$\text{wcol}_k(G) = 1 + \min_L \max_{y \in V} |W_k[G, L, y]|.$$

If we allow paths of any length (but still have restrictions on the position of the internal vertices), we get $S_\infty[G, L, y]$, $W_\infty[G, L, y]$, the *strong ∞ -colouring number* $\text{scol}_\infty(G)$ and the *weak ∞ -colouring number* $\text{wcol}_\infty(G)$.

A rather surprising aspect of the generalised colouring numbers is that they can be seen as gradations between the colouring number and two important graph parameters, namely the tree-width $\text{tw}(G)$ and the tree-depth $\text{td}(G)$ (we shall define *tree-depth* in Chapter 2.)

Proposition 1.2.2.

For every graph G we have:

- (a) $\text{col}(G) = \text{scol}_1(G) \leq \text{scol}_2(G) \leq \cdots \leq \text{scol}_\infty(G) = \text{tw}(G) + 1.$
- (b) $\text{col}(G) = \text{wcol}_1(G) \leq \text{wcol}_2(G) \leq \cdots \leq \text{wcol}_\infty(G) = \text{td}(G).$

The equality $\text{scol}_\infty(G) = \text{tw}(G) + 1$ was first proved in [30]; In Chapter 2, we provide an

alternative proof which is based on the usual definition of tree-width, the one we gave in Section 1.1. The equality $\text{wcol}_\infty(G) = \text{td}(G)$ is proved in [63, Lemma 6.5].

Relations between the two sets of numbers exist as well. Clearly, $\text{col}(G) = \text{scol}_1(G) = \text{wcol}_1(G)$ and $\text{scol}_k(G) \leq \text{wcol}_k(G)$. For the converse, Kierstead and Yang [47] proved that $\text{wcol}_k(G) \leq (\text{scol}_k(G))^k$. Note that this means that if one of the generalised colouring numbers is bounded for a class of graphs (for some k), then so is the other one.

Proposition 1.2.1 tells us that $\text{col}(G)$ provides an upper bound on $\nabla_0(G)$ and vice versa. Zhu [84] proved that a similar relation exists between the generalised colouring numbers and the grad of rank r . In this way, he provided a way of characterising classes with bounded expansion in terms of the weak k -colouring numbers.

Theorem 1.2.3 (Zhu [84]).

A class of graphs \mathcal{K} has bounded expansion if and only if there exist constants c_k , $k = 1, 2, \dots$ such that $\text{wcol}_k(G) \leq c_k$ for all k and all $G \in \mathcal{K}$.

Given the close connection between the weak and the strong colouring numbers, the work of Zhu implicitly provided upper bounds for $\text{scol}_k(G)$ for any K_t -minor free graph G . These bounds, however, grow very fast with k . Grohe, Kreutzer, Rabinovich, Siebertz, and Stavropoulos [30] improved on these upper bounds by providing bounds on $\text{scol}_k(G)$ which are exponential on k . In Chapter 2, we dramatically improve on these upper bounds by providing linear bounds on the strong k -colouring numbers and polynomial bounds on the weak k -colouring numbers of K_t -minor free graphs. For the strong k -colouring numbers, we provide linear upper bounds for H -minor free graphs, for any fixed graph H . In the particular case of planar graphs we can improve our bounds further.

The techniques we develop to obtain these upper bounds are of independent interest. We make use of them again in Chapter 3 in order to find upper bounds for a variant of the generalised colouring numbers. More remarkably, Van den Heuvel and Wood [40] have recently used them to provide impressive results for improper colourings related to Hadwiger's Conjecture.

1.3 Graph powers and exact distance graphs

For a graph $G = (V, E)$ and a positive integer p , the p -th power graph $G^p = (V, E^p)$ of G is the graph with V as its vertex set where E^p contains the edge xy if and only if the distance between x and y satisfies $d_G(x, y) \leq p$. Problems related to the chromatic number $\chi(G^p)$ of power graphs G^p were first considered by Kramer and Kramer [49, 50] in 1969 and have enjoyed significant attention ever since. It is clear that for $p \geq 2$ any power of a star is a clique, and hence there are not many classes of graphs for which $\chi(G^p)$ can be bounded by a constant. An easy argument shows that for a graph G with maximum degree $\Delta(G) \geq 3$ we have

$$\chi(G^p) \leq 1 + \Delta(G^p) \leq 1 + \Delta(G) \cdot \sum_{i=0}^{p-1} (\Delta(G) - 1)^i \in \mathcal{O}(\Delta(G)^p).$$

However, there are many classes of graphs for which it is possible to find much better upper bounds. Such is the case for every class of graphs with bounded degeneracy.

Theorem 1.3.1 (Agnarsson and Halldórsson [3]).

Let k and p be positive integers. There exists a constant $c = c(k, p)$ such that for every k -degenerate graph G we have $\chi(G^p) \leq c \cdot \Delta(G)^{\lfloor p/2 \rfloor}$.

In this result, the exponent on $\Delta(G)$ is best possible (see below). In particular, $\chi(G^2)$ is at most linear in $\Delta(G)$ for planar graphs G . Wegner [82] conjectured that every planar graph G with $\Delta(G) \geq 8$ satisfies $\chi(G^2) \leq \lfloor \frac{3}{2} \Delta(G) \rfloor + 1$, and gave examples that show this bound would be tight. The conjecture has attracted considerable attention since it was stated in 1977. For more on this conjecture we refer the reader to [5, 51].

In [63, Section 11.9], Nešetřil and Ossona de Mendez define the notion of *exact power graph* as follows. Let $G = (V, E)$ be a graph and p a positive integer. The *exact p -power graph* $G^{\natural p}$ of G has V as its vertex set, and xy is an edge in $G^{\natural p}$ if and only if there is in G a path of length p (i.e. with p edges) having x and y as its endpoints (the path need not be induced, nor a shortest path). Similarly, they define the *exact distance- p graph* $G^{\lfloor p \rfloor}$ as the graph with V as its vertex set, and xy as an edge if and only if $d_G(x, y) = p$. Since obviously $E(G^{\lfloor p \rfloor}) \subseteq E(G^{\natural p}) \subseteq E(G^p)$, we have $\chi(G^{\lfloor p \rfloor}) \leq \chi(G^{\natural p}) \leq \chi(G^p)$.

For planar graphs G , Theorem 1.3.1 gives that the exact p -power graphs $G^{\natural p}$ satisfy $\chi(G^{\natural p}) \in \mathcal{O}(\Delta(G)^{\lfloor p/2 \rfloor})$. This result is best possible, as the following examples show. For $k \geq 2$

and $p \geq 4$, let $T_{k, \lfloor p/2 \rfloor}$ be the k -regular tree of radius $\lfloor \frac{1}{2}p \rfloor$ with root v . We say that a vertex y is at level ℓ if $d(v, y) = \ell$. For every edge uv between vertices at levels ℓ and $\ell + 1$ for some $\ell \geq 1$, we do the following: if p is even, then add a path of length $\ell + 1$ between u and v ; if p is odd, then add paths of length $\ell + 1$ and $\ell + 2$ between u and v . Call the resulting graph $G_{k,p}$. It is straightforward to check that $\Delta(G_{k,p}) \leq 2k$ for even p , that $\Delta(G_{k,p}) \leq 3k$ for odd p , and that there is a path of length p between any two vertices at level $\lfloor \frac{1}{2}p \rfloor$. Since there are $k(k-1)^{\lfloor p/2 \rfloor - 1}$ vertices at level $\lfloor \frac{1}{2}p \rfloor$, this immediately means that $\chi(G_{k,p}^{\lfloor p/2 \rfloor}) \geq k(k-1)^{\lfloor p/2 \rfloor - 1} \in \Omega(\Delta(G_{k,p})^{\lfloor p/2 \rfloor})$.

Surprisingly, for exact distance graphs, the situation is quite different.

Theorem 1.3.2.

- (a) *Let p be an odd positive integer. Then there exists a constant $c = c(p)$ such that for every planar graph G we have $\chi(G^{\lfloor p/2 \rfloor}) \leq c$.*
- (b) *Let p be an even positive integer. Then there exists a constant $c' = c'(p)$ such that for every planar graph G we have $\chi(G^{\lfloor p/2 \rfloor}) \leq c' \cdot \Delta(G)$.*

The results in Theorem 1.3.2 are actually special cases of the following more general results.

Theorem 1.3.3.

Let \mathcal{K} be a class of graphs with bounded expansion.

- (a) *Let p be an odd positive integer. Then there exists a constant $C = C(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ we have $\chi(G^{\lfloor p/2 \rfloor}) \leq C$.*
- (b) *Let p be an even positive integer. Then there exists a constant $C' = C'(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ we have $\chi(G^{\lfloor p/2 \rfloor}) \leq C' \cdot \Delta(G)$.*

Theorem 1.3.3 (a) was first proved by Nešetřil and Ossona de Mendez [63, Theorem 11.8]. In Chapter 3 we give two proofs of Theorem 1.3.3 (a). Both proofs are considerably shorter and provide better bounds than the original proof of Nešetřil and Ossona de Mendez. One gives us a best possible result in terms of the generalised colouring numbers, while the other allows us to obtain even better upper bounds for $\chi(G^{\lfloor p/2 \rfloor})$ in various classes. In that same chapter, we also prove Theorem 1.3.3 (b), which is new, as far as we are aware.

As we showed above, if we consider exact powers instead of exact distance graphs, then we need to use bounds involving $\Delta(G)$ if we want to bound $\chi(G^{\lfloor p/2 \rfloor})$, even for odd p and if G is

planar. However, by adding the condition that G has sufficiently large *odd girth* (length of a shortest odd cycle), $\chi(G^{\natural p})$ can be bounded without reference to $\Delta(G)$, for odd p . It follows from Theorem 1.3.3 (a) that this is possible if the odd girth is at least $2p + 1$. This is because odd girth at least $2p + 1$ guarantees that if there is a path of length p between u and v , then any shortest uv -path has odd length. With some more care we can reprove the following.

Theorem 1.3.4 (Nešetřil and Ossona de Mendez [63, Theorem 11.7]).

Let \mathcal{K} be a class of graphs with bounded expansion and let p be an odd positive integer. Then there exists a constant $M = M(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ with odd girth at least $p + 1$ we have $\chi(G^{\natural p}) \leq M$.

Theorem 1.3.2 (a) (and, hence, its general version Theorem 1.3.3 (a)) is quite surprising, since the exact distance graphs $G^{[i]}$ of a planar graph G can be very dense. To see this, for $i \geq 2$ let L_i be obtained from the complete graph K_4 by subdividing each edge $i - 1$ times (i.e. by replacing each edge by a path of length i). For $k \geq 1$, form $L_{i,k}$ by adding four sets of k new vertices to L_i and joining all k vertices in the same set to one of the vertices of degree three in L_i . See Figure 1.1 for a sketch of $L_{1,k}$.

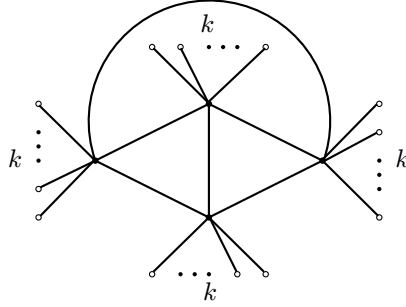


Figure 1.1: A graph $L_{1,k}$ such that $L_{1,k}^{[3]}$ has edge density approximately $3/4$.

It is easy to check that $L_{i,k}$ is a planar graph with $4 + 6(i - 1) + 4k$ vertices, while $L_{i,k}^{[\natural(i+2)]}$ has $6k^2$ edges. So for fixed i and large k , the graph $L_{i,k}^{[\natural(i+2)]}$ has approximately $3/4$ times the number of edges of the complete graph on the same number of vertices. Apart from having unbounded density, the graphs $L_{i,k}^{[\natural(i+2)]}$ have unbounded colouring number (and even unbounded *list chromatic number*) since $L_{i,k}^{[\natural(i+2)]}$ contains a complete bipartite graph $K_{k,k}$ as a subgraph. This makes the fact that these graphs have bounded chromatic number even more surprising.

It is interesting to see what actual upper and lower bounds we can get for the chromatic numbers of $G^{[p]}$ for G from some specific classes of graphs and for specific values of (odd) p . Using the proof in [63], it follows that for $p = 3$ and for planar graphs G we can get the upper bound $\chi(G^{[3]}) \leq 5 \cdot 2^{20,971,522}$ (we give a detailed account of this in Chapter 3). On the other hand, [63, Exercise 11.4] gives an example of a planar graph G with $\chi(G^{[3]}) = 6$.

One way in which we reprove Theorem 1.3.3 (a) is by means of the following result, which also gives us Theorem 1.3.3 (b).

Theorem 1.3.5.

- (a) For every odd positive integer p and every graph G we have $\chi(G^{[p]}) \leq \text{wcol}_{2p-1}(G)$.
- (b) For every even positive integer p and every graph G we have $\chi(G^{[p]}) \leq \text{wcol}_{2p}(G) \cdot \Delta(G)$.

Theorem 1.3.5 (a) already gives a much smaller upper bound for $\chi(G^{[3]})$ for planar graph G . By a more careful analysis, we can reduce that upper bound even further, obtaining that $\chi(G^{[3]}) \leq 143$ for every planar graph G . We also managed to increase the lower bound, although by one only: from 6 to 7. Details can be found in Chapter 3.

Also in Chapter 3, we obtain explicit upper bounds for $\chi(G^{[p]})$ when G excludes a fixed complete graph as a minor, has bounded tree-width, or has bounded genus. Among these results, is the following.

Theorem 1.3.6.

- (a) Let p be an odd integer. For every graph G with tree-width at most t we have $\chi(G^{[p]}) \leq t \cdot \binom{p+t-1}{t} + 1 \in \mathcal{O}(p^{t-1})$.
- (b) Let p be an even integer. For every graph G with tree-width at most t we have $\chi(G^{[p]}) \leq \left(t \cdot \binom{p+t}{t} + 1 \right) \cdot \Delta(G) \in \mathcal{O}(p^t \cdot \Delta(G))$.

In Chapter 4 we focus on a very important type of classes with bounded tree-width and we greatly improve Theorem 1.3.6 for this particular case. A graph G is *chordal* if every cycle of G of length at least 4 has a chord, i.e. if every induced cycle is a triangle. The *clique number* $\omega(G)$ of a graph G is the largest size of a subset of $V(G)$ which induces a complete graph. It is well known [70] that the tree-width of a graph is the smallest integer t such that G is a subgraph of a chordal graph with clique number $t + 1$. In Section 4.4 we prove the following result.

Theorem 1.3.7.

Let G be a chordal graph with clique number $t \geq 2$.

- (a) For every odd integer $p \geq 3$ we have $\chi(G^{[p]}) \leq \binom{t}{2} \cdot (p+1) \in \mathcal{O}(p)$.
- (b) For every even integer $p \geq 2$ we have $\chi(G^{[p]}) \leq \binom{t}{2} \cdot \Delta(G) \cdot (p+1) \in \mathcal{O}(p \cdot \Delta(G))$.

Due to a recent result of Bousquet, Esperet, Harutyunyan, de Joannis de Verclos [13] we known that this linear dependency on p is very close to having the right order. Unfortunately, although every graph of tree-width t is a subgraph of a chordal graph of clique number $t+1$, Theorem 1.3.7 does not extend to all graphs of tree-width at most t . This is because, if $p \geq 2$, it is possible for a subgraph H of a graph G to satisfy $\chi(H^{[p]}) > \chi(G^{[p]})$. However, we prove, also in Chapter 4, that an upper bound on $\chi(G^{[p]})$ for all maximal planar graphs G implies an upper bound on $\chi(H^{[p]})$ for every planar graph, maximal or not.

Proposition 1.3.8.

Let H be a planar graph. There is an edge-maximal planar graph G such that H is a subgraph of G and $\chi(H^{[p]}) \leq \chi(G^{[p]})$ for every positive integer p .

This result might be useful to further improve our upper bounds on $\chi(G^{[3]})$ when G is planar.

1.4 Summary

We conclude this introduction, with a summary of each of the remaining chapters.

In Chapter 2, we study the generalised colouring numbers of classes excluding a fixed minor. We start by providing a new proof of the fact that $\text{scol}_\infty(G) = \text{tw}(G) + 1$ for every graph G . We then introduce *flat decompositions* and a notion of *width* associated to these. A simple proof gives us Lemma 2.3.3, a promising tool which lies at the basis of many results throughout Chapters 2 and 3. We provide upper bounds on the generalised colouring numbers of graphs allowing flat decompositions of small width. This allows us to provide linear bounds on the strong k -colouring numbers and polynomial bounds on the weak k -colouring numbers of K_t -minor free graphs. With a more careful choice of decomposition, we obtain even better bounds for planar graphs and graphs with any fixed genus. This chapter is based on published work [39].

In Chapter 3, we study the chromatic numbers of exact distance graphs and other related graphs such as exact powers. We start by proving that every graph G satisfies $\chi(G^{[p]}) \leq \text{wcol}_{2p-1}(G)$ when p is odd, and $\chi(G^{[p]}) \leq \text{wcol}_{2p}(G) \cdot \Delta(G)$ when p is even. We then prove that there is a function f such that every graph satisfies $\chi(G^{[p]}) \leq f(\text{wcol}_p(G))$, and we give this function explicitly. This result is best possible, as $\chi(G^{[p]})$ cannot be bounded by a function solely of $\text{wcol}_{p-1}(G)$. We prove that if G excludes small cycles of odd length as subgraphs, then the chromatic numbers of the exact powers of G satisfy $\chi(G^{[p]}) \leq f(\text{wcol}_p(G))$ for the same function f as above. We then use a variant of the generalised colouring numbers to find explicit upper bounds for $\chi(G^{[p]})$ when G has bounded tree-width, excludes a fixed complete graph as a minor, is planar, or has bounded genus. We give lower bounds for $\chi(G^{[3]})$ when G is planar or excludes a fixed complete minor. The chapter concludes with a series of lower bounds for the chromatic number of what we call *weak distance graphs* of graphs excluding a fixed complete minor. This chapter is based on work accepted for publication [37].

In Chapter 4, we also study the chromatic number of exact distance graphs, and by restricting our scope only to chordal graphs we manage to obtain improved bounds. Our approach is natural when considering exact distance graphs and has been used successfully in tackling other problems related to chordal graphs: we fix a vertex and partition the vertex set into “levels” according to the distance to this vertex. By studying the types of paths of length p that can exist between vertices in the same level, we prove that if G is a chordal graph with clique number $t \geq 2$, then $\chi(G^{[p]})$ is linear on p and quadratic on t . We end the chapter by proving that an upper bound on $\chi(G^{[p]})$ for all maximal planar graphs G implies an upper bound on $\chi(H^{[p]})$ for every planar graph, maximal or not (this result generalises to classes with bounded genus). This chapter is based on work submitted for publication [68].

In Chapter 5, we introduce a (rather natural) notion of *addition of graph classes*. We then prove that any class of graphs with tree-width at most t and the class of all paths “can be summed” so as to obtain a class of graphs with tree-width at most $t + 2$. Using the notion of *page number*, we also prove that any two (proper) minor closed classes can be summed so as to obtain a class with bounded expansion. We conclude by showing that we can add edges to each graph of a class which excludes a fixed complete topological minor, so as to guarantee that each new graph gets a spanning tree of degree at most 3 and that the resulting class has

bounded expansion. The main tool for proving this last result is a variant of the generalised colouring numbers called *k-admissibility*. This chapter is based on published work [38].

In Chapter 6, we conclude by presenting a collection of open problems which arise from our study of chromatic and structural properties of sparse graph classes.

Chapter 2

Generalised colouring numbers of graphs that exclude a fixed minor

2.1 Introduction

In the same paper in which Nešetřil and Ossona de Mendez introduced classes with bounded expansion [62], they proved that these classes can be characterised in various ways. Two of these characterisations come from the notions of tree-width and tree-depth. The *tree-depth* of a graph, introduced in [61], is an important parameter, particularly within the study of graph homomorphisms, which can be defined as follows.

Let T be a rooted tree, with root $v \in V(T)$. The *height* of a vertex $x \in V(T)$ is $d(v, x) + 1$, and the *height* of T is the maximum height of a vertex in $V(T)$. For $x, y \in V(T)$, we say x is a *descendant* of y if $x \neq y$, and y lies on the unique path between x and the root v . The *closure* $\text{clos}(T)$ of a rooted tree T is the graph with vertex set $V(T)$ and an edge xy if and only if x is a descendant of y , or y a descendant of x , in T . The *tree-depth* $\text{td}(G)$ of a connected graph G is the minimum height of a rooted tree T such that G is a subgraph of $\text{clos}(T)$. The *tree-depth* $\text{td}(G)$ of a graph G is the maximum tree-depth of a connected component of G .

A class \mathcal{K} is said to admit *low tree-width colourings* if, for every $p \geq 1$, there exists an integer $N(p)$ such that any graph $G \in \mathcal{K}$ can be coloured with $N(p)$ colours so that, for every $i \leq p$, each connected component of the subgraph induced by any set of i colours has tree-width at most $i - 1$. Using the structural theory of Robertson and Seymour, DeVos *et al.* [17]

proved that every (proper) minor closed class admits low tree-width colourings. Nešetřil and Ossona de Mendez [62] generalised this result by proving that a class has bounded expansion if and only if it admits low tree-width colourings. They also proved that a class has bounded expansion if and only if it admits *low tree-depth colourings*, which are defined just as low tree-width colourings by replacing the word “tree-width” by the word “tree-depth”.

As we mentioned in the introduction, Zhu [84] proved that classes with bounded expansion can also be characterised in terms of the generalised colouring numbers. In this light, one could expect the generalised colouring numbers to be closely related to the notions of tree-width and tree-depth. This relationship turns out to be remarkably close. As the following proposition tells us, the generalised colouring numbers can be seen as gradations between the colouring number $\text{col}(G)$ and the tree-width and the tree-depth of a graph G .

Proposition 2.1.1 (Proposition 1.2.2).

For every graph G we have:

- (a) $\text{col}(G) = \text{scol}_1(G) \leq \text{scol}_2(G) \leq \cdots \leq \text{scol}_\infty(G) = \text{tw}(G) + 1.$
- (b) $\text{col}(G) = \text{wcol}_1(G) \leq \text{wcol}_2(G) \leq \cdots \leq \text{wcol}_\infty(G) = \text{td}(G).$

The equality $\text{scol}_\infty(G) = \text{tw}(G) + 1$ was first proved in [30]. In this chapter we give an alternative proof. The equality $\text{wcol}_\infty(G) = \text{td}(G)$ was proved in [63, Lemma 6.5].

As tree-width is a fundamental graph parameter with many applications in Structural Graph Theory, most prominently in Robertson and Seymour’s theory of graphs with forbidden minors, it is no wonder that the study of generalised colouring numbers might be of special interest in the context of proper minor closed classes of graphs. As we shall see, excluding a minor indeed allows us to prove strong upper bounds for the generalised colouring numbers.

Using probabilistic arguments inspired by the work of Kierstead and Yang [47] on graphs excluding a fixed topological minor, Zhu [84] was the first to give a non-trivial bound for $\text{scol}_k(G)$ in terms of the densities of depth- r minors of G . For a graph G excluding a complete graph K_t as a minor, Zhu’s bound gives

$$\text{scol}_k(G) \leq 1 + q_k,$$

where q_1 is the maximum average degree of a minor of G , and q_i is inductively defined

by $q_{i+1} = q_1 \cdot q_i^{2i^2}$.

Grohe *et al.* [30] improved Zhu's bounds as follows. They proved that every graph G without a K_t -minor satisfies

$$\text{scol}_k(G) \leq (ckt)^k,$$

for some (small) constant c depending on t .

Our main results are improvements of those bounds for the generalised colouring numbers of graphs excluding a minor.

Theorem 2.1.2.

Let H be a graph and x a vertex of H . Set $h = |E(H - x)|$ and let α be the number of isolated vertices of $H - x$. Then for every graph G that excludes H as a minor, we have

$$\text{scol}_k(G) \leq h \cdot (2k + 1) + \alpha.$$

For classes of graphs that are defined by excluding a complete graph K_t as a minor, we get the following corollary.

Corollary 2.1.3.

For every graph G that excludes the complete graph K_t as a minor, we have

$$\text{scol}_k(G) \leq \binom{t-1}{2} \cdot (2k + 1).$$

For the weak k -colouring numbers we obtain the following bound.

Theorem 2.1.4.

Let $t \geq 4$. For every graph G that excludes K_t as a minor, we have

$$\text{wcol}_k(G) \leq \binom{k+t-2}{t-2} \cdot (t-3)(2k+1) \in \mathcal{O}(k^{t-1}).$$

We refrain from stating a bound on the weak k -colouring numbers in the case that a general graph H is excluded as a minor, for conceptual simplicity. It will be clear from the proof that if a proper subgraph of K_t is excluded, the bounds on $\text{wcol}_k(G)$ can be slightly improved. Those improvements, however, will only be linear in t .

The *acyclic chromatic number* $\chi_a(G)$ of a graph G is the smallest number of colours needed for a proper vertex-colouring of G such that every cycle has at least three colours. The best known upper bound for the acyclic chromatic number of graphs without a K_t -minor is $\mathcal{O}(t^2 \log^2 t)$, implicit in [60]. Kierstead and Yang [47] gave a short proof that $\chi_a(G) \leq \text{scol}_2(G)$. Corollary 2.1.3 shows that for graphs G without a K_t -minor we have $\text{scol}_2(G) \in \mathcal{O}(t^2)$, which immediately gives an improved $\mathcal{O}(t^2)$ upper bound for the acyclic chromatic number of those graphs as well.

In the particular case of graphs with bounded genus, we can improve our bounds further.

Theorem 2.1.5.

For every graph G with genus g , we have $\text{scol}_k(G) \leq (4g + 5)k + 2g + 1$.

In particular, for every planar graph G , we have $\text{scol}_k(G) \leq 5k + 1$.

Theorem 2.1.6.

For every graph G with genus g , we have $\text{wcol}_k(G) \leq \left(2g + \binom{k+2}{2}\right) \cdot (2k+1)$.

In particular, for every planar graph G , we have $\text{wcol}_k(G) \leq \binom{k+2}{2} \cdot (2k+1)$.

For planar graphs, the bound on $\text{scol}_1(G) = \text{col}(G)$ is best possible. Also, for K_t -minor free graphs, one can easily give best possible bounds for $t = 2, 3$ and $k = 1$, as expressed in the following observations.

Proposition 2.1.7.

- (a) *For every graph G that excludes K_2 as a minor, we have $\text{scol}_k(G) = \text{wcol}_k(G) = 1$.*
- (b) *For every graph G that excludes K_3 as a minor, we have $\text{scol}_k(G) \leq 2$ and $\text{wcol}_k(G) \leq k + 1$.*
- (c) *For every graph G that excludes K_t as a minor, $t \geq 4$, we have*

$$\text{scol}_1(G) = \text{wcol}_1(G) \leq (0.64 + o(1)) t \sqrt{\ln t} + 1 \quad \text{as } |V(G)| \rightarrow \infty.$$

Part (a) in the proposition is a triviality. For part (b), note that excluding K_3 as a minor means that G is acyclic, hence a forest, and that in this case it is obvious that $\text{scol}_k(G) \leq 2$ and $\text{wcol}_k(G) \leq k + 1$. Finally, $\text{scol}_1(G) = \text{wcol}_1(G)$ is one more than the degeneracy of G ,

thus part (c) follows from Thomason's bound on the maximum average degree of graphs with no K_t as a minor [80].

Regarding the sharpness on our upper bounds in the results above, we can make the following remarks.

- Lower bounds for the generalised colouring numbers for minor closed classes were given in [30]. In that paper it is shown that for every t and every k there is a graph $G_{t,k}$ of tree-width t that satisfies $\text{scol}_k(G_{t,k}) = t + 1$ and $\text{wcol}_k(G_{t,k}) = \binom{k+t}{t}$. Graphs of tree-width t exclude K_{t+2} as a minor. This shows that our results for classes with excluded minors are optimal up to a factor $(t - 1)(2k + 1)$.

- Since graphs with tree-width 2 are planar, this also shows that there exist planar graphs G with $\text{wcol}_k(G) = \binom{k+2}{2} \in \Omega(k^2)$. Compare this to the upper bound $\text{wcol}_k(G) \in \mathcal{O}(k^3)$ for planar graphs in Theorem 2.1.6.

- It follows from Proposition 2.1.1 (a) that a minor closed class of graphs has uniformly bounded strong colouring numbers if and only if it has bounded tree-width. For classes with unbounded tree-width, such a uniform bound cannot happen. By analysing the shape of admissible paths, it is possible to prove that the planar $k \times k$ grid $G_{k \times k}$ satisfies $\text{scol}_k(G_{k \times k}) \in \Omega(k)$. This, plus Theorem 2.1.5, shows that for planar graphs G , a best possible bound for $\text{scol}_k(G)$ is linear in k .

- It follows from [84, Lemma 3.3] that for 3-regular graphs of high girth the weak k -colouring numbers grow exponentially with k . Hence the polynomial bound for $\text{wcol}_k(G)$ in Theorem 2.1.4 for classes with excluded minors cannot be extended to classes with bounded degree, or to classes with excluded topological minors other than paths and cycles.

The structure of the remainder of this chapter is as follows. In the next section we reprove the connection between the strong colouring numbers and tree-width, as it will be fundamental for our later proofs. In Section 2.3 we introduce *flat decompositions*, which is our main tool in proving our results, and give upper bounds on the generalised colouring numbers of a graph G in terms of the *width* of a flat decomposition of G . In Section 2.4 we prove Theorem 2.1.4, while in Section 2.5 we prove Theorem 2.1.2. In Section 2.6 we prove Theorems 2.1.5 and 2.1.6, which are based on a detailed analysis of the generalised colouring numbers of planar graphs.

2.2 Tree-width and generalised colouring numbers

The concept of tree-width has found many applications in Graph Theory, most remarkably in Robertson and Seymour's theory of graphs with excluded minors and in the study of the complexity of decision problems on graphs. Many problems which are hard to solve in general (such as deciding if a graph G has a cycle which visits all of $V(G)$, that is, a *Hamiltonian cycle*) have polynomial complexity when parametrised by the tree-width of the input graph. A very general theorem due to Courcelle [15] states that every problem definable in monadic second-order logic can be solved in linear time on a class of graphs of bounded tree-width.

It is thus remarkable that the strong k -colouring numbers of a graph G form a gradation between two parameters of such importance as $\deg(G)$ and $\text{tw}(G)$.

In this section we reprove the equality $\text{scol}_\infty(G) = \text{tw}(G) + 1$, which plays a key role in the proofs of this chapter. This was first proved by Grohe *et al.* [30] through the notion of *elimination-width*. Here we prove it directly from the usual definition of tree-width, the one we gave in Section 1.1.

Recall that the bags of a tree-decomposition $(T, (X_y)_{y \in V(T)})$ of a graph G must satisfy:

- (1) $\bigcup_{y \in V(T)} X_y = V(G)$;
- (2) for every edge $uv \in E(G)$, there is a $y \in V(T)$ such that $u, v \in X_y$; and
- (3) if $v \in X_y \cap X_{y'}$ for some $y, y' \in V(T)$, then $v \in X_{y''}$ for all y'' that lie on the unique path between y and y' in T .

We say that a tree-decomposition $(T, (X_y)_{y \in V(T)})$ of width t is *smooth* if all bags have $t + 1$ vertices and adjacent bags share exactly t vertices. It is well known and easy to see that every graph of tree-width t has a smooth tree-decomposition of width t .

Proof of Proposition 2.1.1 (a). It is easy to see that for a graph G with connected components G_1, \dots, G_r , we have $\text{tw}(G) = \max_{1 \leq i \leq r} \text{tw}(G_i)$. Similarly, we have that $\text{scol}_\infty(G) = \max_{1 \leq i \leq r} \text{scol}_\infty(G_i)$. Hence, we can restrict our attention to connected graphs.

We will first prove that every connected graph G satisfies $\text{scol}_\infty(G) \leq \text{tw}(G) + 1$. By definition of $\text{scol}_\infty(G)$, it suffices to find a linear ordering L of $V(G)$ such that every vertex $v \in V(G)$ satisfies $|S_\infty[G, L, v]| \leq \text{tw}(G)$.

Let $(T, (X_y)_{y \in V(T)})$ be a smooth tree-decomposition of width t of G . Let a be a node of T , and assign it as the root of T . We construct a linear ordering L of $V(G)$, from the smallest to the largest vertex, as follows. The smallest vertices in L are those of X_a , and these are ordered in an arbitrary way. We perform a breadth-first search on T . For a vertex $v \in V(G)$, let $b(v) \in V(T)$ be the first node visited by the breadth-first search such that $v \in X_{b(v)}$. For $u, v \notin X_a$, we set $u <_L v$ if $b(u)$ was visited by the breadth-first search before $b(v)$.

Notice that if $v \in V(G)$ belongs to X_a , then it can only strongly ∞ -reach other vertices in X_a , and so v satisfies $|S_\infty[G, L, v]| \leq \text{tw}(G)$. Let us consider, then, $v \notin X_a$. From property (3) of the bags of a tree-decomposition and from our choice of L , it follows that any bag which contains v only contains vertices in $X_{b(v)}$ or vertices which are larger than v in L . This, together with property (2), tells us that if $z \in N(v)$ and $z <_L v$ then we have $z \in X_{b(v)} \setminus \{v\}$. Hence, we can restrict our attention to vertices which are strongly ∞ -reachable from v through paths of length at least 2.

Let $z \in S_\infty[G, L, v]$ be such that there is a path $P = v, z_1, \dots, z_{\ell-1}, z_\ell$ in G with $\ell \geq 2$, $z = z_\ell <_L v$, and $v <_L z_i$ for all $1 \leq i \leq \ell - 1$. By property (2), we know that there must be a node c such that $z_{\ell-1}z_\ell \in X_c$. Since $v <_L z_{\ell-1}$, we know that c is a descendant of $b(v)$ in T . On the other hand, the fact that $z_\ell <_L v$ tells us that the node $b(z_\ell)$ is not a descendant of $b(v)$ in T . Hence, $b(v)$ belongs to the path which joins $b(z_\ell)$ and c in T . Given that z_ℓ belongs to both $X_{b(z_\ell)}$ and X_c , property (3) tells us $z_\ell \in X_{b(v)}$ and, since $z_\ell <_L v$, we get $z = z_\ell \in X_{b(v)} \setminus \{v\}$, as desired.

We now proceed to prove that every connected graph G satisfies $\text{tw}(G) + 1 \leq \text{scol}_\infty(G)$.

For a fixed connected graph G we set $q = \text{scol}_\infty(G)$ and let L be an ordering witnessing $|S_\infty[G, L, v]| \leq q - 1$ for all $v \in V(G)$. We construct a tree-decomposition of G as follows. For each vertex $v \in G$ we create a node $a(v)$, and we let $X_{a(v)} = S_\infty[G, L, v] \cup \{v\}$. If $S_\infty[G, L, v] \neq \emptyset$, we let $\sigma(v)$ be the largest vertex in $S_\infty[G, L, v]$ and we create the edge $a(v)a(\sigma(v))$. Let us see that these nodes together with their corresponding edges form a tree. Since we have assumed that G is connected, it is easy to see that $S_\infty[G, L, v] \neq \emptyset$ for every vertex which is not the smallest vertex with respect to L . Hence, we have created a graph T with $V(G)$ nodes and $V(G) - 1$ edges. Therefore, to prove that T is a tree, it suffices to prove that it is acyclic.

Suppose for a contradiction that there is a cycle C in T and let u be the largest vertex with respect to L such that $a(u)$ belongs to C . Then, we have that there are two vertices $v_1, v_2 <_L u$ such that $a(v_1)a(u), a(v_2)a(u) \in E(T)$. This contradicts our choice of $E(T)$, which guarantees there can be at most one such vertex.

From our choice of bags, it is easy to see that they also satisfy properties (1) and (2), and that the width of the tree-decomposition is at most $\text{scol}_\infty(G) - 1$. Therefore, in order to obtain that $\text{tw}(G) + 1 \leq \text{scol}_\infty(G)$, we only need to prove that the bags we have chosen satisfy property (3).

Let y, y' be two nodes such that $u \in X_y \cap X_{y'}$ for some vertex $u \in V(G)$. We want to prove that $u \in X_{y''}$ for all y'' that lie on the unique path between y and y' in T . Let $v, w \in V(G)$ be such that $y = a(v)$ and $y' = a(w)$. We have that either $u = v$, $u = w$, or $u \in S_\infty[G, L, v] \cap S_\infty[G, L, w]$. In any case, it suffices to show that for every pair $x, x^* \in V(G)$ such that $x \in S_\infty[G, L, x^*]$, we have that x belongs to every bag in the $a(x)a(x^*)$ -path of T .

Since x^* strongly ∞ -reaches $\sigma(x)$ through a path having all internal vertices larger than x^* in L , by means of this path, $\sigma(x^*)$ can strongly ∞ -reach all the vertices in $S_\infty[G, L, x^*] \setminus \{\sigma(x^*)\}$. Let $\sigma_i(x^*) = \sigma(\sigma(\dots\sigma(x^*)\dots))$, where σ is iterated i times. By the same argument, it is easy to see that $\sigma_i(x^*)$ strongly ∞ -reaches all vertices in $S_\infty[G, L, x^*] \setminus \{\sigma(x^*), \sigma_2(x^*), \dots, \sigma_i(x^*)\}$. Let $j(x)$ be such that $x = \sigma_{j(x)}(x^*)$. We have that $x \in S_\infty[G, L, \sigma_i(x^*)]$, for every $0 \leq i \leq j(x) - 1$. By the choice of our bags, this tells us that x belongs to every bag in the $a(x)a(x^*)$ -path of T . \square

In [30], Grohe *et al.* provided a sharp upper bound for the weak colouring numbers $\text{wcol}_k(G)$ of a graph G in terms of its tree-width. The following result is implicit in the proof of [30, Theorem 4.2], and will be very useful for our proofs.

Lemma 2.2.1 (Grohe *et al.* [30]).

Let G be a graph and L a linear ordering of $V(G)$ with $\max_{v \in V(G)} |S_\infty[G, L, v]| \leq t$. For every positive integer k and vertex $v \in V(G)$ we have $|W_k[G, L, v]| \leq \binom{k+t}{t} - 1$.

2.3 Flat decompositions

Our main tool in proving our results will be what we call *flat decompositions*. Roughly speaking, these are ordered vertex partitions where each part has neighbours in only a bounded number of earlier parts and the intersection of each part with the k -neighbourhood of any later part is also bounded. We now give a formal definition of these decompositions.

Let H, H' be vertex disjoint subgraphs of G . We say that H is *connected* to H' if some vertex in H has a neighbour in H' , i.e. if there is an edge $uv \in E(G)$ with $u \in V(H)$ and $v \in V(H')$.

A *decomposition* of a graph G is a sequence $\mathcal{H} = (H_1, \dots, H_\ell)$ of non-empty subgraphs of G such that the vertex sets $V(H_1), \dots, V(H_\ell)$ partition $V(G)$. The decomposition \mathcal{H} is *connected* if each H_i is connected.

For a decomposition (H_1, \dots, H_ℓ) of a graph G and $1 \leq i \leq \ell$, we denote by $G[H_{\geq i}]$ the subgraph of G induced by $\bigcup_{i \leq j \leq \ell} V(H_j)$.

Let G be a graph, let H be a subgraph of G , and let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a function. We say that H *f -spreads on G* if, for every $k \in \mathbb{N}$ and $v \in V(G)$, we have

$$|N_G^k[v] \cap V(H)| \leq f(k).$$

That is, the closed k -neighbourhood of each vertex in G only meets a bounded number of vertices of H .

Definition 2.3.1.

We say a decomposition \mathcal{H} is *f -flat* if each H_i f -spreads on $G[H_{\geq i}]$.

A *flat decomposition* is a decomposition that is f -flat for some function $f: \mathbb{N} \rightarrow \mathbb{N}$.

We shall also define a notion of *width* for decompositions.

Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a decomposition of a graph G , let $1 \leq i < \ell$, and let C be a component of $G[H_{\geq(i+1)}]$. The *separating number* of C is the number $s(C)$ of (distinct) graphs $Q_1, \dots, Q_{s(C)} \in \{H_1, \dots, H_i\}$ such that, for every $1 \leq j \leq s(C)$, there is an edge with an endpoint in C and the other endpoint in Q_j .

Definition 2.3.2.

Let $w_i(\mathcal{H}) = \max s(C)$, where the maximum is taken over all components C of $G[H_{\geq(i+1)}]$.

We define the *width* of \mathcal{H} as $W(\mathcal{H}) = \max_{1 \leq i \leq \ell} w_i(\mathcal{H})$.

Note that the separating number of a component C is independent of the value i such that C is a component of $G[H_{\geq(i+1)}]$. Indeed, let i be minimal such that C is a component of $G[H_{\geq(i+1)}]$. Then for all $t > i$ we have that either H_t is not connected to C , or H_t is a subgraph that contains vertices from C .

We call a path P in G an *optimal path* if P is a shortest path between its endpoints. Optimal paths will play an important role in the analysis of flat decompositions and the generalised colouring numbers. We call a decomposition $\mathcal{H} = (H_1, \dots, H_\ell)$ an *optimal paths decomposition* if H_i is an optimal path in $G[H_{\geq i}]$, for each $1 \leq i \leq \ell$.

A definition similar to optimal paths decompositions was given in [1], where they were called *cop-decompositions*. The name *cop-decomposition* in [1] was inspired by a result of [6], where it was shown that such decompositions of small width exist for classes of graphs that exclude a fixed minor, and these were used to obtain a result for the cops-and-robber game.

Two differences between a cop-decomposition and a connected decomposition are that in a connected decomposition the subgraphs are pairwise disjoint, and that we allow arbitrary connected subgraphs rather than just paths. The first property is extremely useful, as it allows us to contract the subgraphs to find a minor of G with bounded tree-width, as expressed in the following lemma.

Lemma 2.3.3.

Let G be a graph, and let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a connected decomposition of G of width t . By contracting each connected subgraph H_i to a single vertex, we obtain a graph $H = G/\mathcal{H}$ with ℓ vertices and tree-width at most t .

Proof. We identify the vertices of H with the connected subgraphs $\{H_1, \dots, H_\ell\}$. By the contracting operation, two subgraphs H_i, H_j are adjacent in H if there is an edge in G between a vertex of H_i and a vertex of H_j , and there is a path $H_{i(0)}, H_{i(1)}, \dots, H_{i(s)}$ in H , where $H_{i(0)} = H_i$ and $H_{i(s)} = H_j$, if and only if there is a path between some vertex of H_i and some vertex of H_j that uses only vertices of $H_{i(0)}, H_{i(1)}, \dots, H_{i(s)}$, in that order.

Let L be the order on $V(H)$ given by the order of the subgraphs in the connected decomposition. For some arbitrary $H_i \in V(H)$, let us study the set $S_\infty[H, L, H_i]$. This is the set of subgraphs among H_1, \dots, H_{i-1} that are reachable via a path (in H) with internal vertices larger than H_i in L . As each such path corresponds to a path in G as described in the previous paragraph, this is exactly the set of subgraphs in $\{H_1, \dots, H_{i-1}\}$ that are connected in G to the component C of $G[H_{\geq i}]$ that contains H_i . The number of such subgraphs is the separating number of C , which by definition of the width of \mathcal{H} is at most t . This tells us that $|S_\infty[H, L, H_i]| \leq t$ for all $H_i \in V(H)$. This shows that $\text{tw}(H) + 1 = \text{scol}_\infty(H) \leq t + 1$, as required. \square

A fundamental property of optimal paths is that from any vertex v , not many vertices of an optimal path can be reached from v in k steps.

Lemma 2.3.4.

Let v be a vertex of a graph G , and let P be an optimal path in G . Then P contains at most $2k + 1$ vertices of the closed k -neighbourhood of v , that is, $|N^k[v] \cap V(P)| \leq \min\{|V(P)|, 2k + 1\}$.

Proof. Assume $P = v_0, \dots, v_n$ and $|N^k[v] \cap V(P)| > 2k + 1$. Let i be minimal such that $v_i \in N^k[v]$ and let j be maximal such that $v_j \in N^k[v]$. As P is a shortest path, the distance in G between v_i and v_j is $j - i \geq |N^k[v] \cap V(P)| - 1 > 2k$, which contradicts the hypothesis that both v_i and v_j are at distance at most k from v , thus at distance at most $2k$ from each other. \square

From a decomposition (H_1, \dots, H_ℓ) of a graph G , we define a linear order L on $V(G)$ as follows. First choose an arbitrary linear order on the vertices of each subgraph H_i . Now let L be the linear extension of that order, where for $v \in V(H_i)$ and $w \in V(H_j)$ with $i < j$ we define $v <_L w$.

Lemma 2.3.5.

Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a decomposition of a graph G , and let L be a linear order defined from the decomposition. For an integer i , $1 \leq i \leq \ell$, let $G' = G[H_{\geq i}]$. Then, for every $k \in \mathbb{N}$ and every $v \in V(G)$, we have

$$S_k[G, L, v] \cap V(H_i) \subseteq N_{G'}^k[v] \cap V(H_i),$$

$$W_k[G, L, v] \cap V(H_i) \subseteq N_{G'}^k[v] \cap V(H_i).$$

Proof. Let P be a path having as its endpoints $w \in H_i$ and v , and containing a vertex $z \in V(H_1) \cup V(H_2) \cup \dots \cup V(H_{i-1})$. By definition of L , we have $z <_L w$. Hence, v cannot weakly or strongly reach w through P . \square

Now we are in a position to give upper bounds of $\text{scol}_k(G)$ and $\text{wcol}_k(G)$ in terms of the width of a flat decomposition.

Lemma 2.3.6.

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ and let $k, t \in \mathbb{N}$. Let G be a graph that admits an f -flat decomposition of width t . Then we have

$$\text{scol}_k(G) \leq (t + 1) \cdot f(k).$$

Proof. Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be an f -flat decomposition of G of width t , and let L be a linear order defined from the decomposition. Let $v \in V(G)$ be an arbitrary vertex and let q be such that $v \in V(H_{q+1})$. Let C be the component of $G[H_{\geq(q+1)}]$ that contains v , and let Q_1, \dots, Q_m be the subgraphs among H_1, \dots, H_q that have a connection to C . Since \mathcal{H} has width t , we see that $m \leq t$. By definition of L , the vertices in $S_k[G, L, v]$ can only lie on Q_1, \dots, Q_m and on H_{q+1} , hence on at most $t + 1$ subgraphs. For $j = 1, \dots, m$, assume that $Q_j = H_{i_j}$ and let $G'_j = G[H_{\geq i_j}]$. Then by Lemma 2.3.5 we have $S_k[G, L, v] \cap Q_j \subseteq N_{G'_j}^k[v] \cap Q_j$. Since $H_{i_j} = Q_j$ f -spreads on G'_j , we have $|N_{G'_j}^k[v] \cap Q_j| \leq f(k)$. Finally, let $G'_{q+1} = G[H_{\geq q+1}]$. Since, $v \in H_{q+1}$, we have that the open k -neighbourhood of v satisfies $|N_{G'_{q+1}}^k(v) \cap H_{q+1}| \leq f(k) - 1$. The result follows. \square

Lemma 2.3.7.

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ and let $k, t \in \mathbb{N}$. Let G be a graph that admits a connected f -flat decomposition of width t . Then we have

$$\text{wcol}_k(G) \leq \binom{k+t}{t} \cdot f(k).$$

Proof. Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a connected f -flat decomposition of width t , and let L be a linear order defined from it. We contract the subgraphs H_1, \dots, H_ℓ to obtain a graph H of tree-width at most t (Lemma 2.3.3 guarantees we can do this). We identify the vertices of H with the subgraphs H_i . For a vertex $v \in V(G)$, consider the subgraph H_i with $v \in V(H_i)$. By Lemma 2.2.1, the vertex H_i weakly k -reaches at most $\binom{k+t}{t} - 1$ vertices in H that are smaller than H_i in the order on $V(H)$ given by the decomposition. Apart from H_i , these vertices H_j that are weakly k -reachable from H_i in H are the only subgraphs in G that may contain vertices that are weakly k -reachable from v in G . As in the previous proof we can argue that each of these subgraphs contains at most $f(k)$ vertices which are weakly k -reachable from v . Also as in the previous proof, we can argue that H_i contains at most $f(k) - 1$ vertices which are weakly k -reachable from v . The result follows. \square

2.4 The weak k -colouring numbers of graphs excluding a fixed complete minor

In this section we prove Theorem 2.1.4. We will provide a more detailed analysis for the strong k -colouring numbers in the next section.

Theorem (Theorem 2.1.4).

Let $t \geq 4$. For every graph G that excludes K_t as a minor, we have

$$\text{wcol}_k(G) \leq \binom{k+t-2}{t-2} \cdot (t-3)(2k+1) \in \mathcal{O}(k^{t-1}).$$

Theorem 2.1.4 is a direct consequence of Lemma 2.3.7 and of Lemma 2.4.1 below. This lemma states that connected flat decompositions of small width exist for graphs that exclude a fixed complete graph K_t as a minor. This result is inspired by the result on cop-decompositions presented in [6].

Lemma 2.4.1.

Let $t \geq 4$ and let $f: \mathbb{N} \rightarrow \mathbb{N}$ be the function $f(k) = (t-3)(2k+1)$. Let G be a graph that excludes K_t as a minor. Then there exists a connected f -flat decomposition of G of width at most $t-2$.

Proof. Without loss of generality we may assume that G is connected. We will iteratively construct a connected f -flat decomposition H_1, \dots, H_ℓ of G . For all q , $1 \leq q < \ell$, we will maintain the following property. Let C be a component of $G[H_{\geq(q+1)}]$. Then the subgraphs $Q_1, \dots, Q_s \in \{H_1, \dots, H_q\}$ that are connected to C form a minor model of the complete graph K_s , for some $s \leq t-2$. This immediately implies our claim on the width of the decomposition.

To start, we choose an arbitrary vertex $v \in V(G)$ and let H_1 be the connected subgraph $G[v]$. Clearly, H_1 f -spreads on G , and the above property holds (with $s = 1$).

Now assume that for some q , $1 \leq q \leq \ell-1$, the sequence H_1, \dots, H_q has already been constructed. Fix some component C of $G[H_{\geq(q+1)}]$ and assume that the subgraphs $Q_1, \dots, Q_s \in \{H_1, \dots, H_q\}$ that have a connection to C form a minor model of K_s , for some $s \leq t-2$. Because G is connected, we have $s \geq 1$. Let v be a vertex of C that is adjacent to a vertex of Q_1 . Let T be a breadth-first search tree in $G[C]$ with root v . We choose H_{q+1} to be a minimal connected subgraph of T that contains v and that contains for each i , $1 \leq i \leq s$, at least one neighbour of Q_i .

It is easy to see that for every component C' of $G[H_{\geq(q+2)}]$, the subgraphs $Q_1, \dots, Q_{s'} \in \{H_1, \dots, H_{q+1}\}$ that are connected to C' form a minor model of a complete graph $K_{s'}$, for some $s' \leq t-1$. Let us show that in fact we have $s' \leq t-2$. Towards a contradiction, assume that there are $Q_1, \dots, Q_{t-1} \in \{H_1, \dots, H_{q+1}\}$ that have a connection to C' and such that the Q_i form a minor model of K_{t-1} . As each Q_i has a connection to C' , we can contract the whole component C' to find K_t as a minor, a contradiction.

Let us finally show that the decomposition is f -flat. We show that the newly added subgraph H_{q+1} f -spreads on $G[H_{\geq(q+1)}]$. By construction, H_{q+1} is a subtree of T that consists of at most $t-3$ optimal paths in $G[H_{\geq(q+1)}]$ (possibly not disjoint), since T is a breadth-first search tree and v is already a neighbour of Q_1 . Now the claim follows immediately from Lemma 2.3.4. \square

2.5 The strong k -colouring numbers of graphs excluding a fixed minor

For graphs that exclude a complete graph as a minor, we already get, through Lemmas 2.3.6 and 2.4.1, a good bound on the strong k -colouring numbers. However, if a sparse graph is excluded as a minor, we can do much better. In this case we will construct an optimal paths decomposition, where only few paths are separating (in general, each connected subgraph in our proof may subsume many optimal paths).

The proof idea is essentially the same as that for Lemma 2.4.1. We will iteratively construct an optimal paths decomposition (P_1, \dots, P_ℓ) of G such that the components C of $G[P_{\geq(q+1)}]$ are separated by a minor model of a proper subgraph M of $H - x$. To optimise the bounds on the width of the decomposition, we will first try to maximise the number of edges in the subgraph M , before we add more vertices to the model. During the construction we will have to *re-interpret* the separating minor model, as otherwise connections of a vertex model (the subgraph representing a vertex of M) to the component may be lost.

To implement the above mentioned re-interpretation of the minor model it will be more convenient to work with a slightly different (and non-standard) definition of a minor model. Let M be a graph with vertices h_1, \dots, h_m . The graph M is a minor of G if G contains pairwise vertex-disjoint connected subgraphs H_1, \dots, H_m and pairwise internally disjoint paths E_{ij} for $h_i h_j \in E(M)$ that are also internally disjoint from the H_1, \dots, H_m , such that if $e_{ij} = h_i h_j$ is an edge of M , then E_{ij} connects a vertex of H_i with a vertex of H_j . We call the subgraph H_i of G the *model* of h_i in G and the path E_{ij} the *model* of e_{ij} in G .

One can easily see that a graph H is a minor of a graph G according to the definition in Section 1.1 if and only if H is a minor of G according to the definition given above. The reason to introduce paths E_{ij} (rather than edges e_{ij}) is that we want to control the number of vertices in vertex models connected to a component. This is impossible for the connecting paths E_{ij} , so it would be impossible if we let the vertex models grow to encompass the E_{ij} .

Lemma 2.5.1 (following [6]).

Let H be a graph and x a vertex of H . Set $h = |E(H - x)|$, and let α be the number of isolated vertices of $H - x$. Then every graph G that excludes H as a minor admits an optimal paths decomposition of width at most $3h + \alpha$.

Proof. Without loss of generality we may assume that G is connected. Assume $H - x$ has vertices h_1, \dots, h_k , $k = |V(H)| - 1$. For $1 \leq i \leq k$, denote by d_i the degree of h_i in $H - x$.

We will iteratively construct an optimal paths decomposition (P_1, \dots, P_ℓ) of G . For all q , $1 \leq q < \ell$, we will maintain the four properties given below. With each component C of $G[P_{\geq(q+1)}]$ we associate a minor model of a proper subgraph M of $H - x$.

1. For each $h_i \in V(M)$, the model H_i of h_i in G uses vertices of P_1, \dots, P_q only.
2. For each H_i with $h_i \in V(M)$ such that h_i is an isolated vertex in $H - x$, H_i will consist of a single vertex only.

For each H_i with $h_i \in V(M)$ such that h_i is not an isolated vertex in $H - x$, it is possible to place a set of d_i pebbles $\{p_{ij} \mid h_i h_j \in E(H - x)\}$ on the vertices of H_i (with possibly several pebbles on a vertex), in such a way that the pebbles occupy exactly the set of vertices of H_i with a neighbour in C . In particular, each H_i has between 1 and d_i vertices with a neighbour in C .

3. For each edge $e_{ij} = h_i h_j \in E(M)$, the model E_{ij} of e_{ij} in G has the following properties.
 - (a) The endpoints of E_{ij} are the vertices with pebbles p_{ij} in H_i and p_{ji} in H_j .
 - (b) The internal vertices of E_{ij} belong to a single path P_p , where $p \leq q$.
 - (c) Assume E_{ij} has internal vertices in P_p . Let D be the component of $G[P_{\geq p}]$ that contains P_p . Let v_{ij} and v_{ji} be the vertices of H_i and H_j , respectively that are pebbled with p_{ij} and p_{ji} (at the time P_p was defined). Then E_{ij} is an optimal path in $G[D \cup v_{ij}v_{ji}] - e_{ij}$. (This condition is not necessary for the proof of the lemma; it will be used in the proof of Theorem 2.1.2, though.)
4. All vertices on a path of P_1, \dots, P_q that have a connection to C are part of the minor model.

Let us first see that maintaining these properties implies that the optimal paths decomposition has the desired width. By Condition 4, the separating number of the component C is determined by the number of optimal paths that are part of the minor model of M and have a connection to C . To count this number of paths, we count the number m_1 of paths that lie in any vertex model H_i for $h_i \in V(M)$ and have a connection to C , and we count the number m_2 of paths that correspond to the edges e_{ij} of M . By Condition 2, m_1 is at most the number of pebbles in $H - x$ plus the number of isolated vertices of $H - x$. Since the number of pebbles of each model H_i is at most d_i , the number of pebbles is at most the sum of the vertex degrees, and therefore $m_1 \leq 2|E(H - x)| + \alpha$. By Condition 3(b), m_2 is at most $|E(H - x)|$. Finally, since M is a proper subgraph of $H - x$, either $m_1 < 2|E(H - x)| + \alpha$ or $m_2 < |E(H - x)|$ and hence we have $m_1 + m_2 < 3|E(H - x)| + \alpha$.

We show how to construct an optimal paths decomposition with the desired properties. To start, we choose an arbitrary vertex $v \in V(G)$ and let P_1 be the path of length 0 consisting of v only. For every connected component of $G - V(P_1)$, we define M as the single vertex graph K_1 and the model H_1 of this vertex as P_1 . All pebbles are placed on v . As G is connected, we see that Condition 4 is satisfied; all other properties are clearly satisfied.

Now assume that for some q , $1 \leq q \leq \ell - 1$, the sequence P_1, \dots, P_q has already been constructed. Fix some component C of $G[P_{\geq(q+1)}]$ and assume that the pebbled minor model of a proper subgraph $M \subseteq H - x$ with the above properties for C is given. We first find an optimal path P_{q+1} that lies completely inside C and add it to the optimal paths decomposition. The exact choice of P_{q+1} depends on which of the following two cases we are in.

Case 1: There is a pair h_i, h_j of non-adjacent vertices in M such that $h_i h_j \in E(H - x)$. By Condition 2, the pebbles p_{ij} and p_{ji} lie on some vertices v_{ij} of H_i and v_{ji} of H_j , respectively, that have a neighbour in C . Let v_i and v_j be vertices of C with $v_{ij}v_i, v_{ji}v_j \in E(G)$ (possibly $v_i = v_j$) such that the distance between v_i and v_j in C is minimum among all possible neighbours of v_{ij} and v_{ji} in C . We choose P_{q+1} as an arbitrary shortest path in C with endpoints v_i and v_j . We add the edge $h_i h_j$ to M and the path $E_{ij} = v_{ij}v_i + P_{q+1} + v_jv_{ji}$ to the model of M .

Case 2: M is an induced subgraph of $H - x$.

We choose an arbitrary vertex $v \in V(C)$ and define P_{q+1} as the path of length 0 consisting

of v only. We add an isolated vertex h_a to M , for some a with $1 \leq a \leq k$, such that h_a was not already a vertex of M and define $H_a = P_{q+1}$, with any pebbles on v .

Because in both cases the new path P_{q+1} lies completely in C , every other component of $G[P_{\geq(q+1)}]$ (and its respective minor model) is not affected by this path. Therefore, it suffices to show how to find a pebbled minor model with the above properties for every component of $C - V(P_{q+1})$. Let C' be such a component and let M be the proper subgraph of $H - x$ associated with C . We show how to construct from M a graph M' and a corresponding minor model with the appropriate properties for C' . Note that the vertex model H_a added in Case 2 automatically satisfies Conditions 1 and 2.

We iteratively re-establish the properties for the vertex models H_i with $h_i \in V(M)$, in any order. Fix some i with $h_i \in M$ and consider a path E_{ij} such that the vertex $v_{ij} \in V(H_i)$ that is pebbled by p_{ij} has no connection to C' . Let $E_{ij} = w_1, \dots, w_s$, where $w_1 = v_{ij}$. Let a be minimal such that w_a has a connection to C' , or let $a = s - 1$ if no such vertex exists on E_{ij} . We add all vertices w_1, \dots, w_a to H_i . If w_a has a connection to C' , we redefine E_{ij} as the path w_a, \dots, w_s and place the pebble p_{ij} on w_a . If w_a has no connection to C' , we delete the edge $h_i h_j$ from M' . If after fixing every path E_{ij} for H_i in the above way, H_i has no connections to C' , we delete h_i from M' . Otherwise, if there are pebbles that do not lie on a vertex with a connection to C' , we place these pebbles on arbitrary vertices that are occupied by another pebble, that is, that have a connection to C' .

After performing these operations for every H_i , all conditions are satisfied. Condition 2 is re-established for every H_i : if h_i is not removed from M' , then every pebble that lies on a vertex that has no connection to C' is pushed along a path until it lies on a vertex that does have a connection to C' , or finally, if there is no such connection on the path on which end it stands, it is placed at an arbitrary vertex that has a connection to C' . The operations on H_i also re-establish Condition 3(a) for one endpoint of E_{ij} . And after the operations are performed on H_j , Condition 3(a) is re-established for E_{ij} . Furthermore, if C' does not have a connection to a vertex model H_i , it may clearly be removed without violating Condition 4. All other conditions are clearly satisfied.

It remains to show that the graph M for a component C is always a proper subgraph of $H - x$. This however is easy to see. Assume that $M = H - x$ and all conditions are

satisfied. By Condition 2, every H_i , $1 \leq i \leq k$, has a connection to C . Then, by adding C as a subgraph H_{k+1} to the minor model, we find H as a minor, a contradiction. \square

Theorem (Theorem 2.1.2).

Let H be a graph and x a vertex of H . Set $h = |E(H - x)|$, and let α be the number of isolated vertices of $H - x$. Then for every graph G that excludes H as a minor, we have

$$\text{scol}_k(G) \leq h \cdot (2k + 1) + \alpha.$$

Proof. We strengthen the analysis in the proof of Lemma 2.3.6 by taking into account the special properties of the optimal paths decomposition constructed in the proof of Lemma 2.5.1.

Let $\mathcal{P} = (P_1, \dots, P_\ell)$ be an optimal paths decomposition of G that was constructed as in the proof of Lemma 2.5.1, and let L be an order defined from the decomposition. Let $v \in V(G)$ be an arbitrary vertex and let q be such that $v \in V(P_{q+1})$. Let C be the component of $G[P_{\geq(q+1)}]$ that contains v , and let Q_1, \dots, Q_m , $1 \leq m \leq q$, be the paths among P_1, \dots, P_q that have a connection to C . By definition of L , the vertices in $S_k[G, L, v]$ can only lie on Q_1, \dots, Q_m and on P_{q+1} .

In the proof of Lemma 2.5.1, we associated with the component C a pebbled minor model of a proper subgraph M of $H - x$. The paths Q_1, \dots, Q_m were either associated with a vertex model H_i representing a vertex h_i of M , or with a path E_{ij} representing an edge e_{ij} of M . Just as in the proof of Lemma 2.3.6, we can argue that $|S_k[G, L, v] \cap Q_j| \leq \min\{|V(Q_j)|, 2k + 1\}$ for each path Q_j . However, the paths that lie inside a vertex model H_i can have only as many connections to C as there are pebbles on it, since, by Condition 2 of the proof of Lemma 2.5.1, every connection of H_i to C must be pebbled. Let s be the number of paths E_{ij} that have vertices connected to C and in the k -neighbourhood of v . By Condition 3(c) from the proof, for every such path E_{ij} with endpoints v_i and v_j , the pebbles p_{ij} and p_{ji} lie on vertices v_{ij} and v_{ji} such that the path $E'_{ij} = v_{ij}v_i + E_{ij} + v_jv_{ji}$ is optimal. Thus $N_k[v]$ meets only at most h many paths E'_{ij} . It follows from Lemma 2.3.6 that $\text{scol}_k(G) \leq h(2k + 1) + \alpha$. \square

2.6 The generalised colouring numbers of planar graphs

In this section we prove Theorems 2.1.5 and 2.1.6, providing upper bounds for $\text{scol}_k(G)$ and $\text{wcol}_k(G)$ when G is a graph with bounded genus. Since for every genus g there exists a t such that every graph with genus at most g does not contain K_t as a minor, we could use Theorems 2.1.2 to obtain upper bounds for the generalised colouring numbers of such graphs. But the bounds obtained in this section are significantly better.

2.6.1 The weak k -colouring number of planar graphs

By a *maximal planar graph* we mean a (simple) graph that is planar, but where we cannot add any further edges without destroying planarity. It is well known that a maximal planar graph G with $|V(G)| \geq 3$ has a plane embedding (up to the choice of the outer face), which is a triangulation of the plane. We will use that implicitly regularly in what follows.

We start by obtaining an upper bound for $\text{wcol}_k(G)$ that is much smaller than the bound given by Theorem 2.1.2. Our method for doing this again uses optimal paths decompositions. For maximal planar graphs, we will provide optimal paths decompositions of width at most 2. Using Lemma 2.3.7 and the fact that $\text{wcol}_k(G)$ cannot decrease if edges are added, we conclude that $\text{wcol}_k(G) \leq \binom{k+2}{2} \cdot (2k+1) \in \mathcal{O}(k^3)$. In [30], Grohe *et al.* proved that for every k there is a graph $G_{2,k}$ of tree-width 2 such that $\text{wcol}_k(G_{2,k}) = \binom{k+2}{2} \in \Omega(k^2)$. Since graphs with tree-width 2 are planar, this shows that the maximum of $\text{wcol}_k(G)$ for planar graphs is both in $\Omega(k^2)$ and $\mathcal{O}(k^3)$.

Lemma 2.6.1.

Every maximal planar graph G has an optimal paths decomposition of width at most 2.

Proof. Fix a plane embedding of G . Since the proof is otherwise trivial, we assume $|V(G)| \geq 4$.

We will inductively construct an optimal paths decomposition P_1, \dots, P_ℓ such that each component C of $G - \bigcup_{1 \leq j \leq \ell} V(P_j)$ satisfies that the boundary of the region in which C lies is a cycle in G that has its vertices in exactly two paths from P_1, \dots, P_ℓ .

As the first path P_1 , choose an arbitrary edge of the (triangular) outer face, and as P_2 choose the vertex of that triangle that is not contained in P_1 . There is only one connected component in $G - (P_1 \cup P_2)$, and it is in the interior of the cycle which has vertices $V(P_1) \cup V(P_2)$.

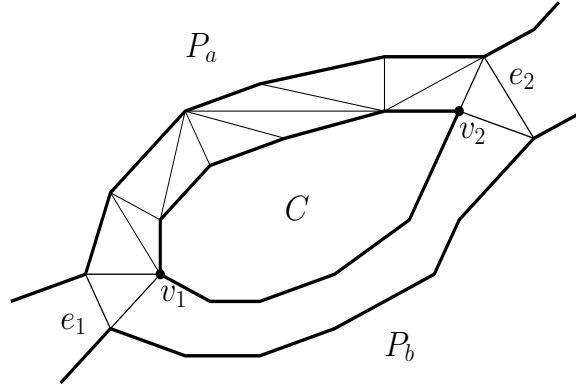


Figure 2.1: The path P_{i+1} is chosen from the vertices of a connected component C .

Now assume that P_1, \dots, P_i have been constructed in the desired way, and choose an arbitrary connected component C of $G - \bigcup_{1 \leq j \leq i} V(P_j)$. Let D be the cycle that forms the boundary of the region in which C lies, and let P_a, P_b , $1 \leq a, b \leq i$, be the paths that contain the vertices of D . Notice that at least one of these paths must have more than one vertex, and let P_a be such a path.

Since P_a and P_b are disjoint and optimal paths, D must contain exactly two edges e_1, e_2 that do not belong to P_a and P_b . (Of course, more than two edges can connect P_a and P_b . But only two of them are on D .) Each of these edges belongs to a triangle in G which is in the interior of D . By definition of D , the triangle that consists of e_1 and a vertex v_1 in the interior of D has the property that v_1 must lie in C . Similarly, the triangle that consists of e_2 and a vertex v_2 in the interior of D has the property that v_2 must lie in C ($v_1 = v_2$ is possible). See Figure 2.1 for a sketch of the situation.

Any path P in C that connects v_1 and v_2 has the property that every vertex of C that is adjacent to P_a is either in P or in the region defined by P_a and P that does not contain P_b . Hence, as a next path P_{i+1} we can take any optimal path in C connecting v_1 and v_2 .

It is clear that any component C' of $G - \bigcup_{1 \leq j \leq i+1} V(P_j)$ that was not already a component of $G - \bigcup_{1 \leq j \leq i} V(P_j)$ is connected to at most two paths from P_a, P_b, P_{i+1} , and no such component is connected to both P_a and P_b . To finish the construction of the decomposition we must prove that such a component C' is connected to exactly two of these three paths. Let us assume that C' lies in the interior of some cycle D' contained in $V(P_a) \cup V(P_{i+1})$. Suppose for a contradiction that D' only has vertices from one of these paths, say from P_a .

But since any cycle contains at least one edge not in P_a and D' has length at least 3, this implies that there is an edge between two non-consecutive vertices of P_a . This contradicts that P_a was chosen as an optimal path. Exactly the same arguments apply when C' lies in the interior of some cycle contained in $V(P_b) \cup V(P_{i+1})$.

The optimal paths decomposition we constructed has width 2, and thus the result follows. \square

Theorem (Theorem 2.1.6).

For every graph G with genus g , we have $\text{wcol}_k(G) \leq \left(2g + \binom{k+2}{2}\right) \cdot (2k+1)$.

In particular, for every planar graph G , we have $\text{wcol}_k(G) \leq \binom{k+2}{2} \cdot (2k+1)$.

Proof. We first prove the bound for planar graphs. According to Lemma 2.6.1, maximal planar graphs have optimal paths decompositions of width at most 2. Using Lemma 2.3.7, we see that any maximal planar graph G satisfies $\text{wcol}_k(G) \leq \binom{k+2}{2} \cdot (2k+1)$. Since $\text{wcol}_k(G)$ cannot decrease when edges are added, we conclude that any planar graph satisfies the same inequality.

It is well known (see e.g. [59, Lemma 4.2.4] or [67]) that for a graph of genus $g > 0$, there exists a non-separating cycle C that consists of two optimal paths such that $G - C$ has genus $g - 1$. We construct a linear order of $V(G)$ by starting with the vertices of such a cycle. We repeat this procedure inductively until all we are left to order are the vertices of a planar graph G' . We have seen that we can order the vertices of G' in such a way that they can weakly k -reach at most $\binom{k+2}{2} \cdot (2k+1)$ vertices in G' . By Lemma 2.3.5 and Lemma 2.3.4 we see that any vertex in the graph can weakly k -reach at most $2g \cdot (2k+1)$ vertices from the cycles we put first in the linear order. The result follows immediately. \square

2.6.2 The strong k -colouring number of planar graphs

From Lemmas 2.6.1 and 2.3.6, we immediately conclude that $\text{scol}_k(G) \leq 3(2k+1)$ if G is planar. This is already an improvement of what we would obtain using Theorem 2.1.2 with the fact that planar graphs do not contain K_5 or $K_{3,3}$ as a minor. Yet we can further improve this by showing that $\text{scol}_k(G) \leq 5k+1$, a bound which is tight for $k = 1$. The method we use

to prove this again uses optimal paths, but differs from the techniques we have used before because we will use sequences of separating paths that are not disjoint.

Let G be a maximal planar graph and fix a plane embedding of G . Let v be any vertex of G and let B be a lexicographic breadth-first search tree of G with root v . For each vertex w , let P_w be the unique path in B from the root v to w .

The following tree-decomposition $(T, (X_t)_{t \in V(T)})$ is a well-known construction that has been used to show that the tree-width of a planar graph is linear in its radius [27].

1. $V(T)$ is the set of faces of G (recall that all these faces are triangles);
2. $E(T)$ contains all pairs $\{t, t'\}$ where the faces t and t' share an edge in G which is not an edge of B ;
3. for each face $t \in V(T)$ with vertices $\{a, b, c\}$, let $X_t = V(P_a) \cup V(P_b) \cup V(P_c)$.

We define a linear order L on the vertices of G as follows. Let t' be the outer face of G , with vertices $\{a, b, c\}$. We pick one of the paths P_a, P_b, P_c , say P_a , arbitrarily as the first path and order its vertices starting from the root v and moving up to a . We pick a second path arbitrarily, say P_b , and order its vertices which have not yet been ordered, starting from the one closest to v and moving up to b . After this, we do the same with the vertices of the third path P_c .

We now pick the outer face as the root of the tree T from the tree-decomposition and perform a depth-first search on T . Each bag X_t contains the union of three paths, but at the moment t is reached by the depth-first search on T , at most one of these paths contains vertices which have not yet been ordered. We order the vertices of such a path starting from the one closest to v and moving up towards the vertex which lies in t .

For $u \in V(G)$, let $f(u)$ be the first face, in the depth-first search traversal of T , for which the bag $X_{f(u)}$ contains u . If u is a vertex for which $f(u)$ is the outer face, then let $C(u)$ be the cycle formed by the three edges in the outer face. Otherwise, if $f(u)$ is not the outer face, then let e_u be the unique edge of $f(u)$ not in B such that the other face containing e_u was found by T before $f(u)$, and let $C(u)$ be the cycle formed in $B + e_u$. Finally, let $O(u)$ be the set of vertices lying in the interior of $C(u)$. See Figure 2.2 for a sketch of the situation.

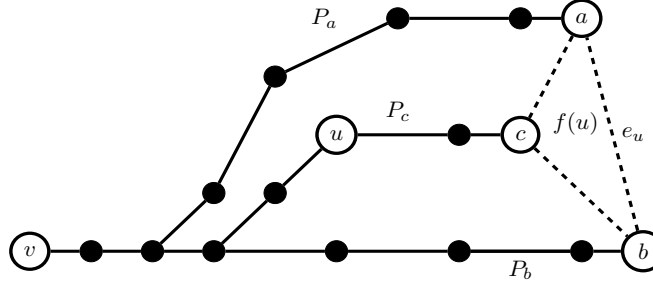


Figure 2.2: Situation for a vertex u such that $f(u)$ is not the outer face. Solid edges represent those in $G[X_{f(u)}] \cap B$. The cycle $C(u)$ is the one contained in $P_a \cup P_b \cup e_u$. The vertices u and c lie in $O(u)$.

The following lemma tells us that if $f(u)$ is not the outer face, then the paths in $X_{f(u)}$ separate u from any other smaller vertex in L .

Lemma 2.6.2.

For all $u \in V(G)$, we have that the vertices of $X_{f(u)}$ are smaller, with respect to L , than all vertices in $O(u) \setminus X_{f(u)}$.

Proof. If $f(u)$ is the outer face, then by the construction of L the vertices of $X_{f(u)}$ are smaller than all other vertices in $V(G)$.

Assume next that $f(u)$ is not the outer face and let $z <_L u$. If $f(z) = f(u)$, then also $X_{f(z)} = X_{f(u)}$ and by the definition of $X_{f(z)}$ we have that z is contained in $X_{f(u)}$. If $f(z) \neq f(u)$, then it must be that $f(z)$ is a face which is encountered before $f(u)$ in the depth-first search of T . We know that $X_{f(z)}$ is the union of three paths. Assume for a contradiction that one of these paths, say P_1 , contains a vertex x in $O(u)$. One of the endpoints of P_1 is in $f(z)$ and therefore cannot be in $O(u)$. The fact that P_1 has both vertices in $O(u)$ and vertices not in $O(u)$, means that there must be a vertex $w \neq v$ of P_1 in $C(u)$. Notice that $C(u) - e_u$ is a subset of two of the paths of $X_{f(u)}$. Therefore, w also belongs to a path P_2 contained in $X_{f(u)}$ that does not have any of its vertices in $O(u)$. That means we have two paths between w and the root v , one is a subpath of P_1 containing x , and the other one is a subpath of P_2 that does not contain x . However, any of the paths that form $X_{f(z)}$ and $X_{f(u)}$ are paths of B , and this means we have found a cycle in B , a contradiction. We conclude that no path of $X_{f(z)}$ contains a vertex of $O(u)$ and so z does not lie in $O(u)$. \square

We will use the ordering L and Lemma 2.6.2 to prove that $\text{scol}_k(G) \leq 5k + 1$ for any planar

graph G . For the purpose of the following proof, it is particularly important that B is a *lexicographic* breadth-first search tree.

Theorem (Theorem 2.1.5).

For every graph G with genus g , we have $\text{scol}_k(G) \leq (4g + 5)k + 2g + 1$.

In particular, for every planar graph G , we have $\text{scol}_k(G) \leq 5k + 1$.

Proof. Also this time we first prove the bound for planar graphs. Since $\text{scol}_k(G)$ cannot decrease when edges are added, we can assume that G is maximal planar. Therefore, we can order its vertices according to a linear order L as defined above.

Fix a vertex $u \in V(G)$ such that $f(u)$ is not the outer face, and let a, b, c be the vertices of $f(u)$. Recall that this means that $X_{f(u)} = V(P_a) \cup V(P_b) \cup V(P_c)$. Choose P_c to be the unique path of $X_{f(u)}$ containing u . Let P_u be the subpath of P_c from u to the root v . Notice that $C(u) - e_u \subseteq P_a \cup P_b$. Then by Lemma 2.6.2, P_a and P_b separate u from all smaller vertices not in $X_{f(u)}$. Therefore, using the definition of the ordering L , we see that all vertices in $V(P_a) \cup V(P_b) \cup V(P_u) \setminus \{u\}$ are smaller than u in L , and that all the vertices in $O(u) \setminus V(P_u)$ are larger than u in L . Hence, we have

$$S_k[G, L, u] \subseteq N_G^k(u) \cap (V(P_a) \cup V(P_b) \cup V(P_u)). \quad (2.1)$$

Since B is a breadth-first search tree, using Lemma 2.3.4, we see that $|N_G^k(u) \cap V(P_a)| \leq 2k + 1$ and $|N_G^k(u) \cap V(P_b)| \leq 2k + 1$. Also, by the definition of L we have $|N_G^k(u) \cap V(P_u)| \leq k$. These inequalities together with (2.1) tell us that $|S_k[G, L, u]| \leq 5k + 2$.

In the remainder of this proof we will show that in fact there are at least 2 fewer vertices in $S_k[G, L, u]$.

We say that the *level* d_u of the vertex u is the distance u has from v , i.e. the height of u in the breadth-first search tree B . For equality to occur in $|N_G^k(u) \cap V(P_a)| \leq 2k + 1$, there must be vertices $z_1, z_2 \in V(P_a)$ in $N_G^k(u)$ such that the level of z_1 in B is $d_u - k$ and the level of z_2 is $d_u + k$. We will show that at most one of z_1 and z_2 can belong to $S_k[G, L, u] \setminus V(P_u)$.

Suppose $z_2 \in S_k[G, L, u]$ and let P_2 be a path from u to z_2 that makes z_2 strongly k -reachable from u . Since z_2 is at level $d_u + k$, P_2 has length k and all of its vertices must be at different levels of B . For any path P with all of its vertices at different levels of B , we will

denote by $P(d)$ the vertex of P at level d . By definition of L , we know that $P_a(d_u + i) <_L z_2$ for all $0 \leq i \leq k - 1$. This, together with the definition of P_2 , tells us that P_2 cannot share any vertex with P_a other than z_2 . Moreover, the edge incident to z_2 in P_2 cannot belong to B , because there already is an edge in $E(P_a) \subseteq E(B)$ joining a vertex at level $d_u + k - 1$ to z_2 . This means that the vertex $P_a(d_u + k - 1)$ was found by the lexicographic breadth-first search B before the vertex $P_2(d_u + k - 1)$. This in its turn implies that $P_a(d_u + k - 2)$ was found by B before $P_2(d_u + k - 2)$. Continuing inductively we find that this is true for every level $d_u + i$, $0 \leq i \leq k - 1$. In a similar way, we can check that this implies that B found the vertex $P_a(d_u - i)$ before the vertex $P_u(d_u - i)$, for $1 \leq i \leq k$, whenever these vertices differ. In particular, z_1 was found before $P_u(d_u - k)$ if $z_1 \notin P_u$.

Let us use this last fact to show that if $z_1 \in S_k[G, L, u]$, then z_1 must also belong to P_u . We do this by assuming that $z_1 \notin V(P_u)$. This tells us that the vertices $P_a(d_u - i)$ and $P_u(d_u - i)$ are distinct for all $0 \leq i \leq k$. As argued before, this means that z_1 was found by B before $P_u(d_u - k)$. This implies that there is no edge between z_1 and $P_u(d_u - k + 1)$, because if it did exist, then the edge joining $P_u(d_u - k)$ and $P_u(d_u - k + 1)$ would not be in B . It follows that any vertex at level $d_u - k + 1$ belonging to $N(z_1)$ was found by B before $P_u(d_u - k + 1)$. By the same argument there is no edge between $N(z_1)$ and $P_u(d_u - k + 2)$. Inductively, we find that for $0 \leq i \leq k - 1$, any vertex at level $d_u - k + i$ belonging to $N_G^i[z_1]$ was found by B before $P_u(d_u - k + i)$, and so there is no edge between $N_G^i[z_1]$ and $P(d_u - k + i + 1)$. But for $i = k - 1$ this means that $u \notin N_G^k[z_1]$ which implies that $z_1 \notin S_k[G, L, u]$. Hence we can conclude that if $z_2 \in S_k[G, L, u]$, then z_1 can only be strongly k -reachable from u if it also belongs to P_u .

Now suppose $z_1 \in S_k[G, L, u] \setminus V(P_u)$, and let P_1 be a path from u to z_1 that makes z_1 strongly k -reachable from u . Since z_1 is at level $d_u - k$, P_1 has length k and all of its vertices are at different levels of B . Let $d_u - j$ be the minimum level of a vertex in $V(P_1) \cap V(P_u)$. Notice that $j < k$, since $z_1 \notin P_u$. Since $E(P_u) \subseteq E(B)$, it is clear that the vertex $P_u(d_u - j - 1)$ was found by B before $P_1(d_u - j - 1)$. This tells us that $P_u(d_u - j - 2)$ was found before $P_1(d_u - j - 2)$. Using induction, we can check that this will also be true for all levels $d_u - i$, $j + 1 \leq i \leq k$. In particular, this means that $P_u(d_u - k)$ was found before z_1 . This implies that the lexicographic search found the vertex $P_u(d_u - i)$ before the vertex $P_a(d_u - i)$, for all

$0 \leq i \leq k$. Hence B found u before $P_a(d_u)$. Now suppose for a contradiction that there is a path P_2 that makes z_2 strongly k -reachable from u . The path P_2 can only intersect $V(P_a)$ at z_2 and, since u was found before $P_a(d_u)$, it must be that $P_2(d_u + i)$ was found by B before $P_a(d_u + i)$, for $1 \leq i \leq k - 1$. Then the edge going from level $d_u + k - 1$ to level $d_u + k$ in P_a does not belong to B . This is a contradiction, given the definition of P_a .

By the analysis above, we have that

$$|(V(P_a) \setminus V(P_u)) \cap S_k[G, L, u]| \leq 2k.$$

In a similar way we can show that $|(V(P_b) \setminus V(P_u)) \cap S_k[G, L, u]| \leq 2k$. Then by (2.1) it follows that $|S_k[G, L, u]| \leq 5k$ for this choice of u .

We still have to do the case that u is a vertex such that $f(u)$ is the outer face. We notice that it might be possible that when u was added to the order L , fewer than two paths reaching $f(u)$ had been ordered. In this case it is clear that $|S_k[G, L, u]| \leq (2k + 1) + k \leq 5k$. If u is on the third chosen path leading from the root to the vertices of $f(u)$, then we can use the arguments above to show that $|S_k[G, L, u]| \leq 5k$.

Having proved the bound on $\text{scol}_k(G)$ for planar graphs, the bound for graphs with genus $g > 0$ can be easily proved following the same procedure as in the proof of Theorem 2.1.6 in the previous subsection. \square

Chapter 3

Chromatic Numbers of Exact Distance Graphs

3.1 Introduction

In this chapter we give upper and lower bounds on the chromatic number of exact distance graphs and other variants of graph powers. The upper bounds will be given in terms of the generalised colouring numbers of the underlying graphs. Using the techniques of the previous chapter, we will be able to give explicit upper bounds for many classes of graphs. For many of these classes our bounds represent a dramatic improvement on existing ones. Moreover, for even distance, our results are new.

In [63, Section 11.9], Nešetřil and Ossona de Mendez introduced the notion of *exact power graph* as follows. Let $G = (V, E)$ be a graph and p a positive integer. The *exact p -power graph* $G^{\natural p}$ has V as its vertex set, and xy is an edge in $G^{\natural p}$ if and only if there is in G a path of length p (i.e. with p edges) having x and y as its endpoints (the path need not be induced, nor a shortest path). Similarly, they define the *exact distance- p graph* $G^{[p]}$ as the graph with V as its vertex set, and xy as an edge if and only if $d_G(x, y) = p$. It is easy to see that, for a graph G , its exact distance graphs, exact powers and powers satisfy $E(G^{[p]}) \subseteq E(G^{\natural p}) \subseteq E(G^p)$, and so we have $\chi(G^{[p]}) \leq \chi(G^{\natural p}) \leq \chi(G^p)$.

Theorem 1.3.1, which was proved by Agnarsson and Halldórsson [3], tells us that the p -power graphs G^p of any planar graph G satisfy $\chi(G^p) \in \mathcal{O}(\Delta(G)^{\lfloor p/2 \rfloor})$. These bounds

actually hold for any fixed class with bounded expansion, and they have the right order in p , even for the class of trees. Since $\chi(G^{\natural p}) \leq \chi(G^p)$, these upper bounds also hold for exact p -powers and, in Chapter 1, we saw that they have the right order, even for the class of planar graphs. In this light, it is surprising how far these bounds are from having the right order when applied to exact distance- p graphs, as the following result attests.

Theorem 3.1.1 (Theorem 1.3.3).

Let \mathcal{K} be a class of graphs with bounded expansion.

- (a) *Let p be an odd positive integer. Then there exists a constant $C = C(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ we have $\chi(G^{\natural p}) \leq C$.*
- (b) *Let p be an even positive integer. Then there exists a constant $C' = C'(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ we have $\chi(G^{\natural p}) \leq C' \cdot \Delta(G)$.*

In this chapter we will give two proofs of Theorem 3.1.1 (a). Both proofs are considerably shorter and provide better bounds than the original proof of Nešetřil and Ossona de Mendez [63, Theorem 11.8]. We also prove Theorem 3.1.1 (b), which is new, as far as we are aware.

Theorem 3.1.1 (a) is surprising, not just because it gives bounds with a very different behaviour from those of Theorem 1.3.1. In Section 1.3, we describe a family of planar graphs $L_{1,k}$, $k \geq 1$, such that the corresponding family of exact distance-3 graphs has unbounded edge density and unbounded colouring number (see Figure 1.1 for an illustration). Nevertheless, Theorem 3.1.1 (a) tells us that there is a constant upper bound on $\chi(L_{1,k}^{\natural 3})$.

The bounds on $\chi(G^{\natural p})$ given in Theorem 1.3.1 (which are exponential on $\Delta(G)$) have the right order in p when G belongs to some fixed class with bounded expansion. However, for odd p , we obtain upper bounds on $\chi(G^{\natural p})$ which are independent of $\Delta(G)$ if we add the condition that G has sufficiently large *odd girth* (length of a shortest odd cycle). It follows from Theorem 3.1.1 (a) that this is possible if the odd girth is at least $2p + 1$. This is because odd girth at least $2p + 1$ guarantees that if there is a path of length p between u and v , then any shortest uv -path has odd length. With some more care we can reprove the following.

Theorem 3.1.2 (Nešetřil and Ossona de Mendez [63, Theorem 11.7]).

Let \mathcal{K} be a class of graphs with bounded expansion and let p be an odd positive integer. Then there exists a constant $M = M(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ with odd girth at least $p + 1$ we have $\chi(G^{\lfloor p \rfloor}) \leq M$.

We prove Theorems 3.1.1 and 3.1.2 by relating, for every graph G , $\chi(G^{\lfloor p \rfloor})$ and $\chi(G^{\lfloor p \rfloor})$ to the weak colouring numbers of G . Before stating our main results in full generality, let us look at some of the bounds that were known for the chromatic number of exact distance graphs. It is particularly illuminating to see what actual upper bounds we can get for the chromatic numbers of $G^{\lfloor p \rfloor}$ for G from some specific classes of graphs and for specific values of p .

The upper bounds obtained by Nešetřil and Ossona de Mendez in their proof of Theorem 3.1.1 (a) are very large, even for $p = 3$. Their proof relies on the concept of *p-centred colourings* of graphs. A (proper) colouring of a graph G is a *p-centred colouring* if for each connected induced subgraph H of G , either one colour appears exactly once on H or H gets at least p colours. The following is what is proved in [63].

Theorem 3.1.3 (Nešetřil and Ossona de Mendez [63]).

*Let p be an odd positive integer. If a graph G has a *p-centred colouring* that uses at most $N = N(p)$ colours, then $\chi(G^{\lfloor p \rfloor}) \leq N2^{N^2}$.*

Given a graph G , the *star chromatic number* $\chi_s(G)$ is the smallest number of colours needed to properly colour G such that every two colours induce a star forest (a forest where every component is isomorphic to a *star* $K_{1,m}$). It is easy to see that a colouring of a graph is 3-centred if and only if every two colours induce a star forest. Albertson *et al.* [4] showed that the star chromatic number of planar graphs is at most 20. This means that the upper bound on $\chi(G^{\lfloor 3 \rfloor})$ for planar graphs given by Theorem 3.1.3 is $5 \cdot 2^{20,971,522}$.

An alternative bound can be obtained from Theorem 3.1.3 using the following result.

Theorem 3.1.4 (Zhu [84]).

*Every graph G has a *p-centred colouring* that uses at most $\text{wcol}_{2p-2}(G)$ colours.*

Corollary 3.1.5.

Let p be an odd positive integer and G a graph. Setting $W = \text{wcol}_{2p-2}(G)$ we have $\chi(G^{\lfloor p \rfloor}) \leq W2^{W^2}$.

As far as we are aware, the best upper bound known for $\text{wcol}_2(G)$ for G planar is given by Theorem 2.1.6. So we have $\text{wcol}_2(G) \leq 30$ for planar graphs, which, when combined with Corollary 3.1.5, unfortunately gives a worse bound for $\chi(G^{\lfloor \frac{1}{2}p \rfloor})$ for planar graphs than the ones obtained earlier.

More recently, Stavropoulos [78] provided a great improvement on the bounds of Corollary 3.1.5.

Theorem 3.1.6 (Stavropoulos [78]).

Let $p \geq 3$ be an odd positive integer. For every graph G we have $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq \text{wcol}_{2p-3}(G)2^{\text{wcol}_{2p-3}(G)}$.

This result, together with Theorem 2.1.6, tells us that every planar graph G satisfies $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq 70 \cdot 2^{70}$.

One of the main results of this chapter is the following theorem. Notice that while the upper bounds of Lemma 3.1.6 are exponential in the weak colouring numbers, these new upper bounds are linear.

Theorem 3.1.7 (Theorem 1.3.5).

- (a) *For every odd positive integer p and every graph G we have $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq \text{wcol}_{2p-1}(G)$.*
- (b) *For every even positive integer p and every graph G we have $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq \text{wcol}_{2p}(G) \cdot \Delta(G)$.*

Together with Theorem 2.1.6, this result tells us that every planar graph G satisfies $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq 231$. Later in this chapter we will show that this bound can be lowered further to 143.

The other main result of this chapter is the following.

Theorem 3.1.8.

Let p be an odd positive integer and G a graph. Set $q = \text{wcol}_p(G)$.

- (a) *We have $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq (\lfloor \frac{1}{2}p \rfloor + 2)^q$.*
- (b) *If G has odd girth at least $p+1$, then $\chi(G^{\lfloor \frac{1}{2}p \rfloor}) \leq (\lfloor \frac{1}{2}p \rfloor + 2)^q$.*

We give the proofs of Theorems 3.1.7 and 3.1.8 in the next section.

The results in Theorem 3.1.8 are best possible in the sense that they give upper bounds of $\chi(G^{\lfloor \frac{1}{2}p \rfloor})$ and $\chi(G^{\lfloor \frac{1}{2}p \rfloor})$ that depend on $\text{wcol}_p(G)$ only, whereas no such results are possible

that depend on $\text{wcol}_k(G)$ with $k < p$. To see this, for $n, p \geq 2$ let $A_{n,p}$ be the $(p-1)$ -subdivision of the complete graph K_n (that is, the graph formed by replacing the edges of K_n by paths of length p). Then we obviously have $\chi(A_{n,p}^{[p]}) = n$. On the other hand we have $\text{wcol}_{p-1}(A_{n,p}) \leq p+1$. To verify this, order the vertices of $A_{n,p}$ as follows. First order the branch vertices (the vertices in the original clique), and then order the subdivision vertices in any way. Clearly, each branch vertex will not weakly $(p-1)$ -reach any other vertex. An internal vertex of a subdivided edge can only weakly $(p-1)$ -reach the other p vertices on the path that replaced the edge (including the two end-vertices of the path). So for fixed odd $p \geq 3$ we cannot bound $\chi(A_{n,p}^{[p]})$ by an expression that involves $\text{wcol}_{p-1}(A_{n,p})$ only.

The bound on the odd girth in Theorem 3.1.8 (b) is also best possible. To show this, for $k, p \geq 3$ let $B_{k,p}$ be formed by taking the path P_{p-1} of length $p-2$, and adding k new vertices that are adjacent to both end-vertices of P_{p-1} only. It is clear that if p is odd, then $B_{k,p}$ has odd girth p . Since between any of the k extra vertices there is a path of length p , we have $\chi(B_{k,p}^{[p]}) \geq k$. The ordering obtained by taking the two end-vertices of P_{p-1} first, and then ordering the other vertices in any way, shows that $\text{wcol}_p(B_{k,p}) \leq p-1$. So for fixed odd $p \geq 3$ we cannot bound $\chi(B_{k,p}^{[p]})$ by an expression that involves $\text{wcol}_p(B_{k,p})$ only.

Nešetřil and Ossona de Mendez [63, Section 11.9.3] give examples that even if we replace “there exists a path of length p between x and y ” by “there exists an *induced* path of length p between x and y ” in the definition of $G^{[p]}$, it is not possible to reduce the bound on the odd girth in Theorem 3.1.8 (b).

The proof of Theorem 3.1.8 actually proves a stronger result. For two graphs $G = (V, E)$ and $G' = (V, E')$ on the same vertex set, define $G \cup G' = (V, E \cup E')$. Then the upper bound in both parts of Theorem 3.1.8 holds for $\chi(G^{[p1]} \cup G^{[p3]} \cup \dots \cup G^{[pp]})$ and $\chi(G^{[p1]} \cup G^{[p3]} \cup \dots \cup G^{[pp]})$, respectively.

A natural question is if for even p we can generalise the bound in Theorem 3.1.7 (b) by a similar bound $\chi(G^{[p2]} \cup G^{[p4]} \cup \dots \cup G^{[pp]}) \leq C \cdot \Delta(G)$, where C depends on the generalised colouring numbers. But this is not possible. Let $T_{\Delta,2}$ be the Δ -regular tree of radius 2. Then we have $\text{wcol}_1(T_{\Delta,2}) = 1$ and $\text{wcol}_k(T_{\Delta,2}) = 2$ for all $k \geq 2$. It is easy to check that $\chi(T_{\Delta,2}^{[p2]}) = \chi(T_{\Delta,2}^{[p4]}) = \Delta$, but $\chi(T_{\Delta,2}^{[p2]} \cup T_{\Delta,2}^{[p4]}) = \Delta(\Delta-1) + 1$. These examples generalise to larger distances.

Let $G = (V, E)$ be a graph and let $x, y \in V$. We say x and y have *weak distance* p if there is a path P of length p between x and y such that any shorter path P' between these vertices is internally vertex disjoint from P (i.e. $V(P) \cap V(P') = \{x, y\}$). We define the *weak distance- p graph* $G^{\flat p}$ as the graph with vertex set V , and having xy as an edge if and only if x and y have weak distance p in G . Note that two vertices in G may have different weak distances between them but, if p is fixed, $G^{\flat p}$ is well-defined. Observe that for any graph G the edge set of $G^{\flat p}$ contains the edge set of $G^{\lfloor \frac{p}{2} \rfloor}$ and that, therefore, $\chi(G^{\lfloor \frac{p}{2} \rfloor}) \leq \chi(G^{\flat p})$. Nevertheless, the proof of Theorem 3.1.8 actually tells us that the upper bound of part (a) holds for $\chi(G^{\flat p})$.

While our main results are upper bounds for the chromatic number of exact distance graphs we also improve on existing lower bounds. Nešetřil and Ossona de Mendez [63, Exercise 11.4] provided a planar graph H such that $\chi(H^{\lfloor \frac{3}{2} \rfloor}) = 6$ (see also [64]). In Section 3.4 we provide the following lower bounds. A graph is *outerplanar* if it can be embedded in the plane in such a way that all its vertices lie in the outer face.

Theorem 3.1.9.

There exist graphs G_4 and G_5 such that we have the following.

- (a) *The graph G_4 is outerplanar and satisfies $\chi(G_4^{\lfloor \frac{3}{2} \rfloor}) = 5$.*
- (b) *The graph G_5 is planar and satisfies $\chi(G_5^{\lfloor \frac{3}{2} \rfloor}) = 7$.*

In Section 3.3 of this chapter we prove that $\chi(G^{\lfloor \frac{3}{2} \rfloor}) \leq 13$ if G is outerplanar. Therefore, this class of graphs is an example in which our lower and upper bounds are close. Also in Section 3.3 we find upper bounds for the chromatic number of exact distance graphs of graphs excluding a fixed complete graph as a minor.

The remainder of this chapter is organised as follows. In the next section we prove our main results, Theorems 3.1.7 and 3.1.8. We use the results from that section in Section 3.3 to find explicit upper bounds for the chromatic number of exact distance graphs for some specific classes of graphs, including graphs with bounded genus, graphs with bounded tree-width, and graphs without a specified complete minor. In Section 3.4 we construct graphs which provide lower bounds for the chromatic number of exact distance-3 graphs of graphs from some minor closed classes. We conclude by giving, in Section 3.5, lower bounds for the chromatic number of weak distance- p graphs of graphs excluding a fixed complete graph as a minor.

3.2 Exact distance graphs and generalised colouring numbers

3.2.1 Proof of Theorem 3.1.7

For later use, we actually prove a slightly stronger result, which involves a more technical variant of the generalised colouring numbers. Let $G = (V, E)$ be a graph, L a linear ordering of V , and k a positive integer. For a vertex $y \in V$, let $D_k[G, L, y]$ be the set of vertices x such that there is a xy -path $P_x = z_0, \dots, z_s$, with $x = z_0$, $y = z_s$, of length $s \leq k$, such that x is the minimum vertex in P_x with respect to L , and such that $y \leq_L z_i$ for $\lfloor \frac{1}{2}k \rfloor + 1 \leq i \leq s$. Thus, unlike the paths considered for $S_k[G, L, y]$ and $W_k[G, L, y]$, the paths considered for $D_k[G, L, y]$ have different restrictions depending on their length. If a vertex $x \in D_k[G, L, y]$ is reached through a path P of length at most $\lfloor \frac{1}{2}k \rfloor$, then P only needs to satisfy that x is minimum. If P has length $\lfloor \frac{1}{2}k \rfloor + 1$ or more, then some of the vertices closest to y in P must be larger than y in L .

We define the *distance- k -colouring number* $\text{dcol}_k(G)$ of a graph G as follows:

$$\text{dcol}_k(G) = 1 + \min_L \max_{y \in V} |D_k[G, L, y]|.$$

Since $S_k[G, L, y] \subseteq D_k[G, L, y] \subseteq W_k[G, L, y]$ for every ordering L , distance k and vertex y , we obtain $\text{scol}_k(G) \leq \text{dcol}_k(G) \leq \text{wcol}_k(G)$. On the other hand, we also have $W_{\lfloor k/2 \rfloor + 1}[G, L, y] \subseteq D_k[G, L, y]$, which implies that $\text{wcol}_{\lfloor k/2 \rfloor + 1}(G) \leq \text{dcol}_k(G)$.

We will prove the following sharpening of Theorem 3.1.7.

Theorem 3.2.1.

- (a) For every odd positive integer p and every graph G we have $\chi(G^{\lfloor kp \rfloor}) \leq \text{dcol}_{2p-1}(G)$.
- (b) For every even positive integer p and every graph G we have $\chi(G^{\lfloor kp \rfloor}) \leq \text{dcol}_{2p}(G) \cdot \Delta(G)$.

Proof. (a) For an odd positive integer p and graph $G = (V, E)$, set $q = \text{dcol}_{2p-1}(G)$ and let L be an ordering of V that witnesses $\max_{y \in V} |D_{2p-1}[G, L, y]| = q - 1$. Moving along the ordering L we assign to each vertex $y \in V$ a colour $a(y) \in [q]$ that is different from $a(x)$ for all $x \in D_{2p-1}[G, L, y]$. Next, define $\mu(y)$ as the minimum vertex with respect to L of the vertices in $N^{\lfloor p/2 \rfloor}[y]$, and define $h : V \rightarrow [q]$ by $h(y) = a(\mu(y))$. We claim that h is a q -colouring of $G^{\lfloor kp \rfloor}$.

Consider an edge $uv \in E(G^{\lfloor p \rfloor})$. So there exists a path $P = z_0, z_1, \dots, z_p$ with $z_0 = u$ and $z_p = v$. Clearly, $N^{\lfloor p/2 \rfloor}[u] \cap N^{\lfloor p/2 \rfloor}[v] = \emptyset$, and hence $\mu(u) \neq \mu(v)$. Without loss of generality, assume $\mu(u) <_L \mu(v)$. Since $\mu(u), z_{\lfloor p/2 \rfloor} \in N^{\lfloor p/2 \rfloor}[u]$, there exists a path S_1 between $\mu(u)$ and $z_{\lfloor p/2 \rfloor}$ of length at most $2\lfloor \frac{1}{2}p \rfloor = p - 1$ such that $V(S_1) \subseteq N^{\lfloor p/2 \rfloor}[u]$. Similarly, there exists a path S_2 between $z_{\lfloor p/2 \rfloor + 1}$ and $\mu(v)$ of length at most $p - 1$ such that $V(S_2) \subseteq N^{\lfloor p/2 \rfloor}[v]$. Since $N^{\lfloor p/2 \rfloor}[u] \cap N^{\lfloor p/2 \rfloor}[v] = \emptyset$ and $z_{\lfloor p/2 \rfloor} z_{\lfloor p/2 \rfloor + 1} \in E$, we can combine these paths to a path S between $\mu(u)$ and $\mu(v)$ of length at most $2p - 1$.

Note that if we write $S = w_0, w_1, \dots, w_t$ with $w_0 = \mu(u)$ and $w_t = \mu(v)$, then the vertices w_i for $\lfloor \frac{1}{2}k \rfloor + 1 \leq i \leq t$ all lie on S_2 , hence are in $N^{\lfloor p/2 \rfloor}[v]$. Since $\mu(v)$ is the minimum vertex in $N^{\lfloor p/2 \rfloor}[v]$, we have $\mu(v) \leq_L w_i$ for those w_i . Thus S witnesses that $\mu(u) \in D_{2p-1}[G, L, \mu(v)]$. We conclude that $h(u) = a(\mu(u)) \neq a(\mu(v)) = h(v)$, as required.

(b) For an even positive integer p and graph $G = (V, E)$, set $q = \text{dcol}_{2p}(G)$ and let L' be an ordering of V that witnesses $\max_{y \in V} |D_{2p}[G, L', y]| = q - 1$. Moving along the ordering L' we assign to each vertex $y \in V$ a colour $a(y) \in [q]$ that is different from $a(x)$ for all $x \in D_{2p}[G, L', y]$. Additionally, for each vertex y , choose an injective function $c_y : N(y) \rightarrow [\Delta(G)]$.

Next, define $\mu(y)$ as the minimum vertex with respect to L' of the vertices in $N^{p/2}[y]$. We also choose an arbitrary vertex in $N(\mu(y)) \cap N^{p/2-1}(y)$; call it $\beta(y)$. To each vertex y we assign as its colour the pair $(a(\mu(y)), c_{\mu(y)}(\beta(y)))$. It is clear that this colouring uses at most $q \cdot \Delta(G)$ colours, and we claim that it is a proper colouring of $G^{\lfloor p \rfloor}$.

Consider an edge $uv \in E(G^{\lfloor p \rfloor})$. First suppose that $\mu(u) \neq \mu(v)$. Then we can follow the proof of part (a) to conclude that $a(\mu(u)) \neq a(\mu(v))$, and hence the colours of u and v differ in the first coordinate.

So we are left with the case $\mu(u) = \mu(v)$. Since $d_G(u, v) = p$, we have that $\mu(v) \in N^{p/2}(u) \cap N^{p/2}(v)$, while $N^{p/2-1}(u) \cap N^{p/2-1}(v) = \emptyset$. This means that $\beta(u) \neq \beta(v)$. Together with the fact that $\beta(u), \beta(v) \in N(\mu(v))$, we obtain that $c_{\mu(v)}(\beta(u)) \neq c_{\mu(v)}(\beta(v))$. This gives that the colours of u and v differ in the second coordinate, which completes the proof. \square

3.2.2 Proof of Theorem 3.1.8

In the proof of Theorem 3.1.8 we use the following lemmas.

Lemma 3.2.2.

Let G be a graph and L a linear ordering of V . Let x, y, z be distinct vertices in G . If x is weakly k -reachable from y , and z is weakly ℓ -reachable from y , then x is weakly $(k+\ell)$ -reachable from z or z is weakly $(k+\ell)$ -reachable from x .

Proof. Since x is weakly k -reachable from y , there is a path $x, v_1, v_2, \dots, v_{r-1}, y$ of length $r \leq k$ for which all internal vertices v_i satisfy $x <_L v_i$. Also, since z is weakly ℓ -reachable from y , there is a path $y, u_1, u_2, \dots, u_{s-1}, z$ of length $s \leq \ell$ for which all internal vertices u_j satisfy $z <_L u_j$. Then, if $x <_L z$, there is an xz -path of length at most $k + \ell$ with all internal vertices greater than x in L ; hence, x is weakly $(k + \ell)$ -reachable from z . Similarly, if $z <_L x$, then z is weakly $(k + \ell)$ -reachable from x . \square

Lemma 3.2.3.

Let p be a positive integer and G a graph with odd girth at least $p + 1$

- (a) Every closed walk of odd length has length at least $p + 1$.
- (b) Let x, y be different vertices and W a walk between x and y of length $r \leq p$. Then there exists a path between x and y of length $s \leq r$ such that s and r have the same parity.

Proof. The proof of (a) is straightforward, since a closed walk of odd length contains a cycle of odd length. For (b), let $W = w_0, \dots, w_r$, with $x = w_0$ and $y = w_r$. If W itself is not a path, then some vertex z appears more than once in W . The part of W between the first and last appearances of z is a closed walk W' of length $t \leq r$. Using (a) we obtain that t must be even. Hence, if we remove W' from W , we get a shorter walk between x and y of length $r - t \equiv r \pmod{2}$. Hence, if we do not immediately obtain a path, we can repeat this procedure inductively until we obtain an xy -path with the desired property. \square

Proof of Theorem 3.1.8.

For both parts of the theorem we use the same colouring. Let L be an ordering of V such that $\max_{y \in V} |W_p[G, L, y]| = q - 1$. We first create an auxiliary colouring $a(y) \in [q]$ by moving along the ordering L , and assigning to each vertex $y \in V$ a colour $a(y) \in [q]$ that is different from $a(x)$ for all $x \in W_p[G, L, y]$. Next, for a vertex $x \in W_{\lfloor p/2 \rfloor}[G, L, y]$, let $d'_y(x)$ be the minimum integer k such that x is weakly k -reachable from y , and set $d'_y(y) = 0$.

Define the function $b_y : [q] \rightarrow [\lfloor \frac{1}{2}p \rfloor] \cup \{-1, 0\}$ as follows. For a colour $c \in [q]$, let

$$b_y(c) = \begin{cases} d'_y(x), & \text{if there exists an } x \in W_{\lfloor p/2 \rfloor}[G, L, y] \cup \{y\} \text{ with } a(x) = c; \\ -1, & \text{otherwise.} \end{cases}$$

By Lemma 3.2.2 and the definition of $a(x)$, we see that if $x \in W_{\lfloor p/2 \rfloor}[G, L, y] \cup \{y\}$ satisfies $a(x) = c$, then x is the only vertex in $W_{\lfloor p/2 \rfloor}[G, L, y] \cup \{y\}$ with colour c . That implies that b_y is well defined.

The number of possible functions $b_y : [q] \rightarrow [\lfloor \frac{1}{2}p \rfloor] \cup \{-1, 0\}$ is $(\lfloor \frac{1}{2}p \rfloor + 2)^q$. We will prove that labelling each vertex $y \in V$ with b_y gives a proper colouring for the graphs and situations described in parts (a) and (b) of the theorem. It is more convenient to do part (b) first.

(b) Consider two vertices u, v for which there exists a path of length p between u and v . Without loss of generality we assume $u <_L v$. If u is weakly p -reachable from v in L , then we know that $a(u) \neq a(v)$, and hence $b_u(a(u)) = 0 \neq b_v(a(u))$.

So we are left with the case in which u is not weakly p -reachable from v in L . Let k be the length of the shortest odd-length path between u and v . We obviously have $k \leq p$. Because u is not weakly p -reachable from v in L , we also have $k \neq 1$, hence $k \geq 3$. Let $P = z_0, z_1, z_2, \dots, z_{k-1}, z_k$ be a path of length k between $u = z_0$ and $v = z_k$. Let z_ℓ be the vertex of P that is minimum with respect to the ordering L . Since $u <_L v$, we get that $z_\ell \neq v$, and, since u is not weakly p -reachable from v , we see that $z_\ell \neq u$. Therefore, z_ℓ is weakly l -reachable from u and weakly $(k - \ell)$ -reachable from v .

First consider the case that $\ell < k - \ell$. Then $\ell < \frac{1}{2}k$. We want to prove that $d'_u(z_\ell) = \ell$. For this, assume that $d'_u(z_\ell) = m < \ell$. Hence there is a path A between u and z_ℓ of length m . If ℓ and m have different parity, then the union of A and the path z_0, z_1, \dots, z_ℓ gives a closed walk of odd length $m + \ell < 2\ell < k \leq p$, which contradicts Lemma 3.2.3 (a). So m and ℓ have the same parity. Now if we replace in the path P the part z_0, z_1, \dots, z_ℓ with A , we get a walk between u and v of length $k - \ell + m < k$, hence with odd length. By Lemma 3.2.3 (b), this walk contains a path between u and v of odd length at most $k - \ell + m < k$, which contradicts the choice of P .

So we know that $d'_u(z_\ell) = \ell$. Notice that since there is a path of length $k - \ell$ between z_ℓ and v , we have that $d'_v(z_\ell) \leq k - \ell \leq p - \ell$. Since $\ell < \frac{1}{2}k \leq \frac{1}{2}p$, we have that $z_\ell \in$

$W_{\lfloor p/2 \rfloor}[G, L, u]$, and hence $b_u(a(z_\ell)) = \ell$.

Now consider a vertex $x \in W_{\lfloor p/2 \rfloor}[G, L, v]$ with $d'_v(x) = \ell$. We first prove that $x \neq z_\ell$. For suppose this is not the case, then there is a path from v to z_ℓ of length ℓ . Together with the part of $z_\ell, z_{\ell+1}, \dots, z_k = v$ from the path P , this gives a closed walk of length $k \leq p$. Since k is odd, this contradicts Lemma 3.2.3 (a).

Since $d'_v(x) = \ell$, $d'_v(z_\ell) \leq p - \ell$ and $x \neq z_\ell$, by Lemma 3.2.2 we get that x is weakly p -reachable from z_ℓ or z_ℓ is weakly p -reachable from x . This gives $a(x) \neq a(z_\ell)$, which implies, by choice of x , that $b_v(a(z_\ell)) \neq \ell$.

If $k - \ell < \ell$, we can prove in a similar way that $b_u \neq b_v$, which completes the proof of part (b) of the theorem.

(a) This time we consider two vertices u, v that have distance k in G , for some odd integer $k \leq p$. (To prove the statement, it would be enough to prove the case $k = p$, but we prefer to give the proof of a more general statement.) We can more or less follow the proof of part (b) above, working with a shortest path $P = z_0, z_1, z_2, \dots, z_{k-1}, z_k$ between $u = z_0$ and $v = z_k$.

Since P is a shortest path, we immediately get that $d'_u(z_\ell) = d_G(u, z_\ell) = \ell$ and $d'_v(z_\ell) = d_G(v, z_\ell) = p - \ell$. This also means that $x \neq z_\ell$, since $d_G(v, x) \leq d'_v(x) = \ell < p - \ell$. For the remainder, the proofs are exactly the same. \square

The proofs of Theorem 3.1.8 (a) and (b) above give results that are stronger than the statements in the theorem. We already discussed in Subsection 1.3 that in fact we prove upper bounds on $\chi(G^{\lfloor p/2 \rfloor} \cup G^{\lfloor p/2 \rfloor+1} \cup \dots \cup G^{\lfloor p/2 \rfloor+p})$ and $\chi(G^{\lfloor p/2 \rfloor} \cup G^{\lfloor p/2 \rfloor+1} \cup \dots \cup G^{\lfloor p/2 \rfloor+p})$. We also mentioned that in part (a) we could replace exact distance- p graphs by weak distance- p graphs. This is equivalent to replacing the condition that we add an edge uv to $G^{\lfloor p/2 \rfloor}$ if $d_G(u, v) = p$, i.e. “there is a shortest path of length p between u and v ”, by the weaker condition “there is a path P of length p between u and v such that any shorter path between those vertices is internally disjoint from P ”.

3.3 Explicit upper bounds on the chromatic number of exact distance graphs

In this section we use Theorem 3.2.1 (a) to find explicit upper bounds for the chromatic number of exact distance graphs for certain types of graphs, including planar graphs, graphs with bounded tree-width, and graphs without a fixed complete minor. Obtaining these bounds involves finding upper bounds for the distance- k -colouring numbers $\text{dcol}_k(G)$. More explicitly, we will prove the following results.

Theorem 3.3.1.

For every graph G with genus g , we have $\text{dcol}_k(G) \leq \left(2g + 2 \binom{\lfloor k/2 \rfloor + 2}{2} + 1\right) \cdot (2k + 1)$.

In particular, for every planar graph G , we have $\text{dcol}_k(G) \leq \left(2 \binom{\lfloor k/2 \rfloor + 2}{2} + 1\right) \cdot (2k + 1)$.

Theorem 3.3.2.

For every graph G with tree-width at most t , we have $\text{dcol}_k(G) \leq t \cdot \binom{\lfloor k/2 \rfloor + t}{t} + 1$.

Theorem 3.3.3.

Let $t \geq 4$. For every graph G that excludes K_t as a minor, we have

$$\text{dcol}_k(G) \leq \left((t-2) \binom{\lfloor k/2 \rfloor + t - 2}{t-2} + 1\right) \cdot (t-3)(2k+1).$$

Since outerplanar graphs G have tree-width at most 2, combining Theorems 3.2.1 (a) and 3.3.2 gives $\chi(G^{[\lfloor 3 \rfloor]}) \leq 13$. Similarly, from Theorem 3.3.1 we see that for planar graphs G we have $\chi(G^{[\lfloor 3 \rfloor]}) \leq 143$, while for graphs G embeddable on the torus we have $\chi(G^{[\lfloor 3 \rfloor]}) \leq 165$.

We devote the remainder of this section to proving Theorems 3.3.1, 3.3.2 and 3.3.3. They are based on the methods developed in Chapter 2 to obtain bounds for the generalised colouring numbers.

3.3.1 Graphs with bounded tree-width

In Chapter 2 we obtained upper bounds for the weak colouring numbers of graphs excluding a fixed complete minor and of graphs with bounded genus. But at the base of our argument was

a result of Grohe *et al.* [30] which provides sharp upper bounds for the weak colouring numbers $\text{wcol}_k(G)$ of a graph G in terms of its tree-width. Using Lemma 2.2.1 (which follows from the proof of [30, Theorem 4.2]), we now provide upper bounds on the distance- k -colouring numbers $\text{dcol}_k(G)$ of a graph G in terms of its tree-width.

Proof of Theorem 3.3.2. Since G has tree-width at most t , we have $\text{scol}_\infty(G) \leq t + 1$, and so there is an ordering L of $V(G)$ such that $\max_{y \in V(G)} |S_\infty[G, L, y]| \leq t$. By Lemma 2.2.1 we get that for every positive k and $y \in V(G)$ we have $|W_k[G, L, y]| \leq \binom{k+t}{t} - 1$. Consider $v \in V(G)$, and for each $u \in D_k[G, L, v]$ fix a path $P_u = z_0, \dots, z_s$ with $s \leq k$, $v = z_s$, $u = z_0$, such that u is minimum on P_u , and such that $\sigma(u)$ (which denotes the greatest index with $z_{\sigma(u)} <_L v$) satisfies $0 \leq \sigma(u) \leq \lfloor \frac{1}{2}k \rfloor$. Such a path exists by the definition of $D_k[G, L, v]$. It is clear that subpaths of P_u give that $z_{\sigma(u)}$ is strongly $(s - \sigma(u))$ -reachable from v and u is weakly $\sigma(u)$ -reachable from $z_{\sigma(u)}$. Since $s \leq k$ and $0 \leq \sigma(u) \leq \lfloor \frac{1}{2}k \rfloor$, we obtain that $z_{\sigma(u)}$ is strongly k -reachable from v and u is weakly $\lfloor \frac{1}{2}k \rfloor$ -reachable from $z_{\sigma(u)}$. Together with our choice of L , this gives us the following estimates:

$$|D_k[G, L, v]| \leq |S_k[G, L, v]| \cdot (|W_{\lfloor k/2 \rfloor}[G, L, v]| + 1) \leq t \cdot \binom{\lfloor k/2 \rfloor + t}{t}.$$

The result follows immediately. \square

3.3.2 Graphs with excluded complete minors

In order to provide upper bounds for the generalised colouring numbers for graphs that exclude a fixed minor, we introduced in Chapter 2 the notions of *connected decompositions* and *flat decompositions*. We also introduced a notion of *width* for these decompositions. We will make use of these decompositions as we obtain bounds for the distance- k -colouring numbers.

Lemma 2.3.3 relates the width of a connected decomposition to the tree-width of the minor obtained by contracting each part. The proof of this lemma exemplifies the power of generalised colouring numbers. It gives a short argument that by contracting the subgraphs of a connected decomposition $\mathcal{H} = (H_1, \dots, H_\ell)$ of width t we obtain a graph H which satisfies $\text{scol}_\infty(H) \leq t + 1$. An upper bound on the tree-width then follows by Proposition 2.1.1. Moreover, the proof shows that the ordering L of $V(H)$ obtained by setting $H_i <_L H_j$ if $i < j$

satisfies $\max_{1 \leq i \leq \ell} |S_\infty[H, L, H_i]| \leq t$. Using this property we can prove that if the decomposition from which H was obtained is f -flat, then we can find an upper bound on $\text{dcol}_k(G)$ in terms of $f(k)$.

Lemma 3.3.4.

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ and let t, k be positive integers. For every graph G that admits a connected f -flat decomposition of width at most t we have $\text{dcol}_k(G) \leq \left(t \cdot \binom{\lfloor k/2 \rfloor + t}{t} + 1\right) \cdot f(k)$.

Proof. The proof of this lemma is similar to that of Lemma 2.3.7. Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a connected f -flat decomposition of G of width t . Since \mathcal{H} is connected, we know, by Lemma 2.3.3, that contracting the subgraphs in \mathcal{H} leads to a graph H with tree-width at most t . We identify the vertices of H with the subgraphs H_i , and define a linear ordering L on $V(H)$ by setting $H_i <_L H_j$ if $i < j$. By the proof of Lemma 2.3.3 we get that L satisfies $\max_{1 \leq i \leq \ell} |S_\infty[H, L, H_i]| \leq t$. Using Lemma 2.2.1 this implies that $|W_{\lfloor k/2 \rfloor}[H, L, H_i]| \leq \binom{\lfloor k/2 \rfloor + t}{t} - 1$ for any vertex $H_i \in V(H)$. Arguing as in the proof of Theorem 3.3.2, this means that for every $H_i \in V(H)$ we have $|D_k[H, L, H_i]| \leq t \cdot \binom{\lfloor k/2 \rfloor + t}{t}$.

From L we define an ordering L' on $V(G)$ in the following way. For $u \in H_i$ and $v \in H_j$ with $i \neq j$, we let $u <_{L'} v$ if $i < j$. Then, for every $1 \leq i \leq \ell$, we arbitrarily order the vertices of H_i in any order. It is easy to see that any vertex $v \in H_i$ satisfies

$$D_k[G, L', v] \subseteq N^k[v] \cap (H_i \cup \{H_j \mid H_j \in D_k[H, L, H_i]\}).$$

Hence, we have that there are at most $t \cdot \binom{\lfloor k/2 \rfloor + t}{t} + 1$ subgraphs among H_1, \dots, H_ℓ in G that contain vertices from $D_k[G, L', v]$. Since \mathcal{H} is f -flat, we know that the intersection of each of these subgraphs with $N^k[v]$ is at most $f(k)$. Finally, since $D_k[G, L', v]$ is a proper subset of $N^k[v]$ (as $v \notin D_k[G, L', v]$), the result follows. \square

By proving Lemma 2.4.1, we showed that graphs that do not contain a complete graph as a minor have flat decompositions of small width. Combining Lemmas 3.3.4 and 2.4.1 immediately gives Theorem 3.3.3.

Lemma 2.6.1 provides flat decompositions of even smaller width for maximal planar graphs. This lemma allows us, together with Lemma 3.3.4, to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. By combining Lemmas 3.3.4 and 2.6.1, we get the bound on $\text{dcol}_k(G)$ for maximal planar graphs. Since $\text{dcol}_k(G)$ cannot decrease when edges are added, we conclude that any planar graph satisfies the same inequality.

Having proved the bound on $\text{dcol}_k(G)$ for planar graphs, the bound for graphs with genus $g > 0$ can be easily proved following the same procedure as in the proof of Theorem 2.1.6 in Subsection 2.6.1. \square

3.4 Lower bounds on the chromatic number of exact distance-3 graphs

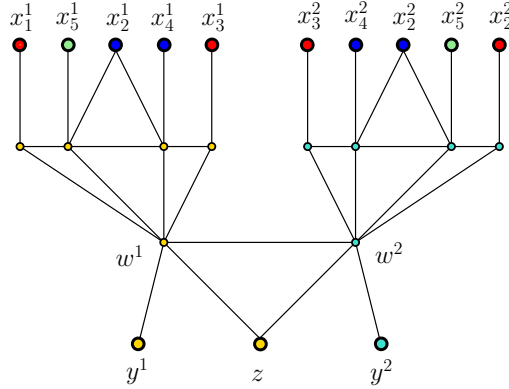


Figure 3.1: An outerplanar graph G_4 with $\chi(G_4^{[3]}) = 5$.

In [63, Exercise 11.4] a planar graph G such that $\chi(G^{[3]}) = 6$ is given (see also [64]). As we will prove below, the outerplanar graph G_4 in Figure 3.1 satisfies $\chi(G_4^{[3]}) = 5$. We will use that graph to construct a planar graph G_5 such that $\chi(G_5^{[3]}) = 7$.

Theorem (Theorem 3.1.9).

There exist graphs G_4 and G_5 such that we have:

- (a) *The graph G_4 is outerplanar and satisfies $\chi(G_4^{[3]}) = 5$.*
- (b) *The graph G_5 is planar and satisfies $\chi(G_5^{[3]}) = 7$.*

Proof. We will prove first that $\chi(G_4^{[3]}) = 5$, using the vertex labelling in Figure 3.1. Consider a proper colouring of $G_4^{[3]}$. Note that $C^1 = x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_1^1$ and $C^2 = x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1^2$ form disjoint 5-cycles $G_4^{[3]}$. Hence, the vertices in $V(C^1) \cup V(C^2)$ need at least 3 colours.

Given that $V(C^1) \cup V(C^2) \subseteq N(z)$ in $G_4^{[\natural 3]}$, if we use more than 3 colours on $V(C^1) \cup V(C^2)$, then we already use at least 5 colours. So assume that the vertices in $V(C^1) \cup V(C^2)$ are coloured with 3 colours only. Since $V(C^i) \subseteq N(y^i)$ in $G_4^{[\natural 3]}$ for $i = 1, 2$, and $y^1 y^2 \in E(G_4^{[\natural 3]})$, we need at least 2 extra colours. So we always use at least 5 colours in a proper colouring of $G_4^{[\natural 3]}$. Figure 3.1 gives a colouring of G_4 with 5 colours which is a proper colouring of $G_4^{[\natural 3]}$. This shows that $\chi(G_4^{[\natural 3]}) = 5$.

Now let F_1 and F_2 be two disjoint copies of G_4 . Let H be a path on 5 vertices, disjoint from F_1 and F_2 , with vertices $y'_1, w'_1, z', w'_2, y'_2$ in that order, together with the edge $w'_1 w'_2$. (This is exactly the graph formed by the vertices $\{y^1, w^1, z, w^2, y^2\}$ in Figure 3.1.) The graph G_5^- has vertex set and edge set:

$$V(G_5^-) = V(F_1) \cup V(F_2) \cup V(H);$$

$$E(G_5^-) = E(F_1) \cup E(F_2) \cup E(H) \cup \{b_1 w'_1 \mid b_1 \in V(F_1)\} \cup \{b_2 w'_2 \mid b_2 \in V(F_2)\}.$$

Finally, the graph G_5 is obtained from G_5^- by subdividing once all the edges of the form $b_1 w'_1$ and $b_2 w'_2$ (replacing each edge by a path of length 2). Since G_4 is outerplanar, it is easy to check that G_5 is planar.

If $u, v \in V(F_1)$ and P is an uv -path in G_5 but $V(P) \not\subseteq V(F_1)$, then $w'_1 \in V(P)$. Thus the length of P is at least 4. We conclude that if two vertices u, v have distance 3 in G_5 , then any shortest uv -path has all its vertices in $V(F_1)$. Therefore, the number of colours needed to colour the vertices of F_1 in $G_5^{[\natural 3]}$ is 5, and the same applies to F_2 . We now can argue as in the proof of $\chi(G_4^{[\natural 3]}) = 5$ above to reach the conclusion $\chi(G_5^{[\natural 3]}) = 7$. \square

Since the graph G_4 in Figure 3.1 is outerplanar, it does not have K_4 as a minor. Also, the graph G_5 we constructed above is planar, so does not have K_5 as a minor. We can iterate the construction to obtain graphs G_t that are K_t -minor free, for $t \geq 4$, and for which $\chi(G_t^{[\natural 3]}) \geq 2(t-2) + 1$. To obtain G_{t+1} from G_t , we take two copies of G_t , one copy of the graph H from above, and add paths of length 2 between all vertices in the first copy of G_t and w'_1 , and between all vertices in the second copy of G_t and w'_2 . It is straightforward to check that if G_t is K_t -minor free, then G_{t+1} is K_{t+1} -minor free, and that $G_{t+1}^{[\natural 3]}$ needs at least 2 more colours than $G_t^{[\natural 3]}$ does.

The property that for $t \geq 5$ there exists graphs G that are K_t -minor free and satisfy $\chi(G^{[t3]}) \geq 2(t-2) + 1$ does not extend to $t = 3$. To see this, note that the only graphs that are K_3 -minor free are acyclic graphs (i.e. forests), which implies they are bipartite. And for bipartite graphs G we have that $G^{[t3]}$ is bipartite as well (in fact, even the exact p -power graph $G^{[tp]}$ is bipartite for every odd p), hence $\chi(G^{[t3]}) \leq 2$.

Notice that one can construct the graph G_4 of Figure 3.1 (and the graphs G_t for $t \geq 4$) by using operations similar to those of used in the Hajós construction [35]. Consider the graph S induced on G_4 by $(N(w^1) \setminus w^2) \cup \{w_1, x_1^1, x_2^2, \dots, x_5^1\}$. The main connected component of the graph $S^{[t3]}$ consists of a cycle and two apex vertices, z and y_1 , that are adjacent to all the vertices in the cycle. One can obtain G_4 by taking two copies of S , identifying the two vertices that correspond to z , and adding an edge between the two vertices that correspond to w_1 . In the exact distance-3 graph, we see that one of the apex vertices has been identified, while those that correspond to y_1 have been joined by an edge. However, the operation of deletion, used in the Hajós construction, is not used in our construction. This is mainly because we want to obtain a graph with chromatic number strictly larger than that of the parts it is formed of.

3.5 Lower bounds on the chromatic number of weak distance- p graphs

In Section 3.2 we implicitly gave upper bounds for the chromatic number of weak distance- p graphs in terms of the generalised colouring numbers. In this section we provide lower bounds for the number of colours needed to colour $G^{[tp]}$ when G is K_t -minor free, for any fixed integer t and any fixed odd p . In fact, we will obtain lower bounds for the clique number of the corresponding weak distance- p graphs.

From the lower bounds we obtained in Section 3.4 for the chromatic number of exact distance-3 graphs, one can easily obtain the same lower bounds for larger (odd) distances by simply adding pendant edges (or induced paths) to some vertices. However, the idea of pendant edges will not work in the case of weak distance- p graphs.

The lower bounds we obtain for weak distance- p graphs grow much faster in t than those

we obtain for exact distance graphs (from linear to factorial growth). In the case of planar graphs, we will obtain a lower bound of 36.

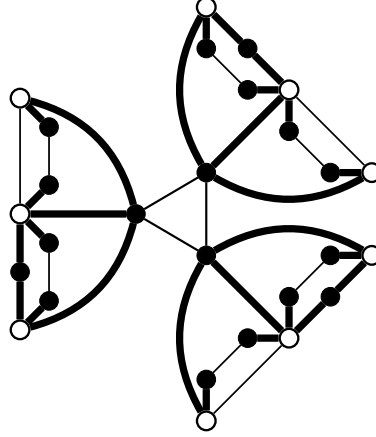


Figure 3.2: Graph $H_{4,3}$. Any pair of white vertices is adjacent in G^{b3} .

Theorem 3.5.1.

Let $t \geq 4$ be a positive integer and $p \geq 3$ an odd integer. There is a K_t -minor free graph $H_{t,p}$ such that $\omega(H_{t,p}^{bp}) \geq \frac{3}{2}(t-1)!$.

Proof. Let $H_{4,3}$ be the graph in Figure 3.2 and let $B(H_{4,3}) \subset V(H_{4,3})$ be the set of white vertices in that figure. (Just as with white and black vertices, the difference between thick and thin edges exclusively aims to partition the edge set in a convenient way.) Clearly, any two vertices in $B(H_{4,3})$ are adjacent in $H_{4,3}^{b3}$, and hence we have $\omega(H_{4,3}^{b3}) \geq 9 = \frac{3}{2} \cdot 3!$.

For $p > 3$ odd, we define $H_{4,p}$ by taking $H_{4,3}$ and subdividing any edge uz with $u \in B(H_{4,3})$ and $z \in V(H_{4,3}) \setminus B(H_{4,3})$ (the thick edges in Figure 3.2) exactly $\frac{1}{2}(p-3)$ times. (Equivalently, every such edge is replaced by a path of length $\frac{1}{2}(p-3) + 1$.) The resulting graph is $H_{4,p}$. Notice that any path of length 3 in $H_{4,3}$ joining two vertices in $B(H_{4,3})$ has exactly two of the subdivided edges. Hence every such path of length 3 is transformed into a path of length $2(\frac{1}{2}(p-3) + 1) + 1 = p$ in $H_{4,p}$. Let $B(H_{4,p}) = B(H_{4,3})$. We have just shown that there is a path of length p in $H_{4,p}$ between any two vertices in $B(H_{4,p})$.

It is easy to see that a path of length less than p in $H_{4,p}$ between vertices in $B(H_{4,p})$ is formed from a path of length at most 2 in $H_{4,3}$. The reader can check that for every pair of white vertices in $H_{4,3}$, there is a path of length three with two thick edges which is disjoint

from all paths of length at most 2. Therefore, the corresponding paths of length p in $H_{4,p}$ will be disjoint from the paths of shorter length which join the same pair of vertices of $B(H_{4,p})$.

We conclude that any two vertices in $B(H_{4,p})$ are adjacent in $H_{4,p}^{b3}$, and so $\omega(H_{4,p}^{bp}) \geq |B(H_{4,p})| = \frac{3}{2} \cdot 3!$, for $p \geq 3$ odd.

Next we notice that the graph $H_{4,3}$ in Figure 3.2 does not contain K_4 as a minor, and therefore $H_{4,p}$ is K_4 -minor free, for $p \geq 3$ odd.

For fixed $p \geq 3$ odd, we construct $H_{t,p}$ recursively from $H_{t-1,p}$ ($t \geq 5$). Let F_1, F_2, \dots, F_{t-1} be $t-1$ disjoint copies of $H_{t-1,p}$, and let v_1, v_2, \dots, v_{t-1} be the vertices of a new K_{t-1} , disjoint from each copy of $H_{t-1,p}$. We subdivide $p-3$ times each edge of K_{t-1} (i.e. each edge is replaced by a path of length $p-2$); the resulting graph we call K_{t-1}^p . Then form the graph $H_{t,p}$ with vertex set and edge set:

$$\begin{aligned} V(H_{t,p}) &= \bigcup_{i=1}^{t-1} V(F_i) \cup V(K_{t-1}^p); \\ E(H_{t,p}) &= \bigcup_{i=1}^{t-1} E(F_i) \cup E(K_{t-1}^p) \cup E', \end{aligned}$$

$$\text{where } E' = \{xv_i \mid x \in B(F_i), i = 1, \dots, t-1\}.$$

For every $1 \leq i \leq t-1$, $H_{t-1,p}[V(F_i) \cup v_i]$ does not contain K_t as a minor since F_i does not contain K_{t-1} as a minor. Moreover, any path joining a vertex from a copy F_i to a vertex from $H_{t,p} \setminus F_i$, has to go through v_i . From this we easily see that $H_{t,p}$ is K_t -minor free.

We define $B(H_{t,p}) = \bigcup_{i=1}^{t-1} B(F_i)$. We will show that the vertices of $B(H_{t,p})$ form a clique in $H_{t,p}^{bp}$. We first see that vertices $u_i \in B(F_i)$ and $u_j \in B(F_j)$, $i \neq j$, are joined by a path of length p of the form u_i, v_i, v_j, u_j , where the edge $v_i v_j$ is replaced by the appropriate subdivision in K_{t-1}^p . We also note that there is no shorter path joining u_i and u_j , so they are adjacent in $H_{t,p}^{bp}$. On the other hand, any pair of vertices $u, z \in B(F_i)$ are adjacent in the graph induced by F_i in $H_{t,p}^{bp}$. The only new path created between them in the construction of $H_{t,p}$ is the path u, v_i, z , which is clearly internally disjoint from any other path joining z and u in F_i . Hence u and z are adjacent in $H_{t,p}^{bp}$. We conclude that any two vertices in $B(H_{t,p})$ are adjacent in $H_{t,p}^{bp}$. By construction, we have that $|B(H_{t,p})| = (t-1)|B(H_{t-1,p})|$ and the result follows. \square

Noticing that $H_{5,3}$ is planar and therefore $H_{5,p}$ is also planar for every odd $p \geq 3$, we obtain the following result.

Corollary 3.5.2.

Let $p \geq 3$ an odd integer. Then there is a planar graph $H_{5,p}$ such that $\omega(H_{5,p}^{\text{bp}}) \geq 36$.

Chapter 4

Colouring exact distance graphs of chordal graphs

4.1 Introduction

As mentioned in Chapter 1, Agnarsson and Halldórsson [3] found upper bounds on the chromatic numbers of graph powers of graphs from many different classes of graphs. They obtained these bounds by parametrising in terms of the degeneracy and, with regard to this parametrisation, their bounds are best possible. We restate here this important result.

Theorem 4.1.1 (Theorem 1.3.1).

Let k and p be positive integers. There exists a constant $c = c(k, p)$ such that for every k -degenerate graph G we have $\chi(G^p) \leq c \cdot \Delta(G)^{\lfloor p/2 \rfloor}$.

For some classes of graphs it is possible to obtain similar bounds without parametrising in terms of the degeneracy. Recall that a graph G is chordal if every cycle of G has a chord, i.e., if every induced cycle is a triangle. In [52], Král' proved that every chordal graph G with maximum degree Δ satisfies $\chi(G^p) \in \mathcal{O}(\sqrt{p}\Delta^{(p+1)/2})$ for even p , and $\chi(G^p) \in \mathcal{O}(\Delta^{(p+1)/2})$ for odd p . Král' also showed that the order of this upper bound for odd p is best possible. It is worth mentioning that, in order to obtain this tight upper bound, Král' gave a simple proof of the already known fact that odd powers of chordal graphs are also chordal [8, 20].

Given that graphs with tree-width at most t have degeneracy at most t , Theorem 4.1.1

gives us an upper bound on $\chi(G^p)$ when G belongs to a graph class with bounded tree-width. Although the tree-width of a graph is usually defined in terms of tree-decompositions, it can also be characterised in terms of chordal graphs, as follows.

Proposition 4.1.2 (Robertson and Seymour [70]).

The tree-width of a graph G is the smallest integer t such that G is a subgraph of a chordal graph with clique number $t + 1$.

Given that $\chi(G^{\lfloor p \rfloor}) \leq \chi(G^p)$ for every graph G and integer p , Theorem 4.1.1 implies upper bounds on the chromatic number of exact distance graphs of graphs with bounded tree-width. However, when considering exact distance graphs, this upper bound is far from best possible. This is attested, for instance, by the following result which is a consequence of Theorems 3.1.7 and 3.3.2.

Theorem 4.1.3 (Theorem 1.3.6).

(a) *Let p be an odd integer. For every graph G with tree-width at most t we have*

$$\chi(G^{\lfloor p \rfloor}) \leq t \cdot \binom{p+t-1}{t} + 1 \in \mathcal{O}(p^{t-1}).$$

(b) *Let p be an even integer. For every graph G with tree-width at most t we have*

$$\chi(G^{\lfloor p \rfloor}) \leq \left(t \cdot \binom{p+t}{t} + 1 \right) \cdot \Delta(G) \in \mathcal{O}(p^t \cdot \Delta(G)).$$

The main result in this chapter is a significant improvement on the bounds of Theorem 4.1.3 for chordal graphs. We prove this result in Section 4.4.

Theorem 4.1.4 (Theorem 1.3.7).

Let G be a chordal graph with clique number $t \geq 2$.

(a) *For every odd integer $p \geq 3$ we have $\chi(G^{\lfloor p \rfloor}) \leq \binom{t}{2} \cdot (p+1) \in \mathcal{O}(p)$.*

(b) *For every even integer $p \geq 2$ we have $\chi(G^{\lfloor p \rfloor}) \leq \binom{t}{2} \cdot \Delta(G) \cdot (p+1) \in \mathcal{O}(p \cdot \Delta(G))$.*

Although Proposition 4.1.2 tells us that every graph of tree-width t is a subgraph of a chordal graph with clique number $t + 1$, Theorem 4.1.4 does not extend to all graphs with tree-width at most t . We shall say more about this at the end of this section. Before that, let us state the full generality of our results and mention yet another direction in which we improve on existing bounds.

For two graphs $G = (V, E)$ and $G' = (V, E')$ on the same vertex set, define $G \cup G' = (V, E \cup E')$. For a fixed positive integer p , Theorem 4.1.4 trivially gives $\chi(G^{[\lfloor p/2 \rfloor]} \cup G^{[\lfloor p/2 \rfloor]} \cup \dots \cup G^{[\lfloor p/2 \rfloor]}) \leq \binom{t}{2}^s \cdot \Delta(G)^q \cdot (p+1)^s$ for any subset $\{p_1, p_2, \dots, p_s\}$ of $[p]$ with q even elements. Notice that if we take $\{p_1, p_2, \dots, p_s\} = [p]$, then we have $G^{[\lfloor p/2 \rfloor]} \cup G^{[\lfloor p/2 \rfloor]} \cup \dots \cup G^{[\lfloor p/2 \rfloor]} = G^p$. Taking a subset of even integers turns out to be quite different from taking a subset of odd integers. For even p , we note that the Δ -regular tree of radius $\lfloor p/2 \rfloor$, $T_{\Delta, p}$, shows that $\chi(T_{\Delta, p}^{[\lfloor p/2 \rfloor]} \cup T_{\Delta, p}^{[\lfloor p/2 \rfloor]} \cup \dots \cup T_{\Delta, p}^{[\lfloor p/2 \rfloor]}) \in \Omega(\Delta^{p/2})$. Hence, the bound of Theorem 4.1.1 gives again the right exponent on $\Delta(G)$. In contrast, we see that for odd p we obtain an upper bound on $\chi(G^{[\lfloor p/2 \rfloor]} \cup G^{[\lfloor p/2 \rfloor]} \cup \dots \cup G^{[\lfloor p/2 \rfloor]})$ which does not depend on $\Delta(G)$. However, these trivial upper bounds stop being linear in p , even if we simply consider $\chi(G^{[\lfloor (p-2)/2 \rfloor]} \cup G^{[\lfloor (p-2)/2 \rfloor]})$.

We prove Theorem 4.1.4 by proving the following stronger result which gives upper bounds on the chromatic number of all these gradations between $G^{[\lfloor p/2 \rfloor]}$ and G^p . For instance, these upper bounds are linear in p if the subsets of $[p]$ considered have bounded size. For the case of chordal graphs, these results greatly improve on the bounds of Theorem 4.1.3. Moreover, they improve on the constant term of Theorem 4.1.1.

Theorem 4.1.5.

Let G be a chordal graph with clique number $t \geq 2$. Let p be a positive integer, $S = \{p_1, p_2, \dots, p_s\} \subseteq [p]$ and q be the number of even integers in S .

- (a) *If $1 \notin S$, then we have $\chi(G^{[\lfloor p/2 \rfloor]} \cup G^{[\lfloor p/2 \rfloor]} \cup \dots \cup G^{[\lfloor p/2 \rfloor]}) \leq \binom{t}{2}^s \cdot \Delta(G)^q \cdot (p+1)$.*
- (b) *If $1 \in S$, then we have $\chi(G^{[\lfloor p/2 \rfloor]} \cup G^{[\lfloor p/2 \rfloor]} \cup \dots \cup G^{[\lfloor p/2 \rfloor]}) \leq t \cdot \binom{t}{2}^{s-1} \cdot \Delta(G)^q \cdot (p+1)$.*

Of course, if $S = \{1\}$ then we have that $\chi(G^{[\lfloor p/2 \rfloor]} \cup G^{[\lfloor p/2 \rfloor]} \cup \dots \cup G^{[\lfloor p/2 \rfloor]}) = \chi(G) = t$, given the well known fact that chordal graphs are perfect and hence satisfy $\chi(G) = \omega(G)$.

We obtain Theorem 4.1.5 by partitioning the graph G into levels. We fix a vertex $x \in V(G)$ and we define the level ℓ as the set of vertices having distance ℓ to x . We bound the number of colours needed to colour one level and then give different colours to levels which are at distance at most p . Apart from being natural in the context of exact distance graphs, this simple levelling argument is regularly used in colouring problems related to perfect graphs. (The real problem is, of course, in the analysis of each level.) Kündgen and Pelsmajer [55] used level partitions of chordal graphs to find an upper bound on the number of colours needed in a nonrepetitive colouring of a graph with tree-width t . More recently, Scott and

Seymour [75] used level partitions to prove a conjecture of Gyárfás stating that there is a function f such that $\chi(G) \leq f(\omega(G))$ for every graph G with no odd hole. In fact, this paper of Scott and Seymour is just the first of a long list of papers which, making extensive use of level partitions, study the relationship between chromatic number, clique number and odd holes.

Together with level partitions, the notion of an adjacent-cliques graph is fundamental in our proof of Theorem 4.1.5. For a graph G and two cliques K and K^* in G , we say that K and K^* are *adjacent* if they are disjoint and there is a pair of vertices $x \in K$, $y \in K^*$ with $xy \in E(G)$. The *adjacent-cliques graph* $AC(G)$ of a graph G has a vertex for each clique of G , and two vertices z and z^* of $AC(G)$ are adjacent if and only if their corresponding cliques in G are adjacent. We prove the following result for the chromatic number of $AC(G)$ when G is chordal.

Theorem 4.1.6.

Let G be a chordal graph with clique number at most t . We have $\chi(AC(G)) \leq \binom{t+1}{2}$.

We denote the line graph of a graph G by $L(G)$. It is easy to see that $AC(G)$ contains G and $L(G)^{[\lfloor 2 \rfloor]}$ as subgraphs. Hence, Theorem 4.1.6 tells us that for all chordal graphs G with clique number t there is an upper bound on $\chi(L(G)^{[\lfloor 2 \rfloor]})$. This is surprising, considering that, even for $t = 2$, $L(G)^{[\lfloor p \rfloor]}$ can have arbitrarily large cliques if p is odd (consider stars and subdivided stars). Whether or not there are upper bounds on $\chi(L(G)^{[\lfloor p \rfloor]})$ for even $p \geq 4$ is an open problem.

As we mentioned before, although every graph of tree-width t is a subgraph of a chordal graph of clique number $t + 1$, Theorems 4.1.4 and 4.1.5 do not extend to all graphs of tree-width at most t . This is because, if $p \geq 2$, it is possible for a subgraph H of a graph G to satisfy $\chi(H^{[\lfloor p \rfloor]}) > \chi(G^{[\lfloor p \rfloor]})$. However, we prove that for any class with bounded genus, any upper bound on $\chi(G^{[\lfloor p \rfloor]})$ for all maximal graphs G in the class implies the same upper bound on $\chi(H^{[\lfloor p \rfloor]})$ for every graph H in the class, maximal or not.

Proposition 4.1.7.

Let H be a graph with genus $g \geq 0$. There is an edge-maximal graph G of genus g such that H is a subgraph of G and $\chi(H^{[\lfloor p \rfloor]}) \leq \chi(G^{[\lfloor p \rfloor]})$ for every positive integer p .

The rest of the chapter is organised as follows. In the next section we study the properties of level partitions of chordal graphs which will be essential for the proof of Theorem 4.1.5. In Section 4.3 we prove Theorem 4.1.6, and in Section 4.4 we complete the proof of Theorem 4.1.5. We conclude by proving Proposition 4.1.7 in Section 4.5.

4.2 Level partitions of chordal graphs

Let G be a graph and x be a fixed vertex of G . For any positive integer ℓ , set $N^\ell(x) = \{v \in V(G) \mid d(v, x) = \ell\}$. We call $N^\ell(x)$ the ℓ -th level of G with respect to x , and if we set $N^0(x) = \{x\}$ we get that these levels partition the connected component of G containing x . We also set $N^{<\ell}(x) = \bigcup_{i < \ell} N^i(x)$ and $N^{>\ell}(x) = \bigcup_{i > \ell} N^i(x)$.

Let G_k , $G_{<k}$ and $G_{>k}$ be the graphs induced by $N^k(x)$, $N^{<k}(x)$ and $N^{>k}(x)$, respectively. Define the ℓ -shadow of a subgraph H of G as the set of vertices in $N^\ell(x)$ which have a neighbour in $V(H)$. We say that G is *shadow complete* (with respect to x) if for every non-negative integer ℓ , the ℓ -shadow of every connected component of $G_{>\ell}$ induces a complete graph.

Using a well-known theorem of Dirac [19] which characterises chordal graphs in terms of their minimal vertex cut sets, Kündgen and Pelsmajer [55] proved that connected chordal graphs are shadow complete with respect to any vertex.

Lemma 4.2.1 (Kündgen and Pelsmajer [55]).

Let G be a connected chordal graph with clique number $t \geq 2$ and let x be any vertex in $V(G)$. Then G is shadow complete with respect to x and every G_ℓ is a chordal graph with clique number strictly smaller than t .

Before we start to see some implications of this lemma, let us state one additional definition. We say that a vertex $v \in N^\ell(x)$, is an *ancestor* (with respect to x) of a vertex $u \in N^m(x)$, $\ell < m$, if there is a path between u and v of length $m - \ell$. Clearly this path has exactly one vertex in each level $N^\ell(x), N^{\ell+1}(x), \dots, N^m(x)$.

The following result follows directly from Lemma 4.2.1.

Corollary 4.2.2.

Let G be a connected chordal graph with clique number $t \geq 2$, $x \in V(G)$, and $u, v \in N^\ell(x)$ for some positive integer ℓ . If u and v are both ancestors of some $y \in N^{>\ell}(x)$, then u and v are neighbours.

With a bit more care we can prove that if two vertices are at the same level ℓ and are at distance p , then their ancestors at level $\ell - \lfloor p/2 \rfloor$ form cliques which either intersect or are adjacent.

Lemma 4.2.3.

Let G be a connected chordal graph with clique number $t \geq 2$, $x \in V(G)$, and $u, v \in N^\ell(x)$ for some positive integer ℓ . Suppose $d(u, v) = p \geq 2$ and let K_u, K_v be the complete graphs induced by the set of ancestors of u and v in $N^{\ell - \lfloor p/2 \rfloor}(x)$, respectively. We have that

- (a) if p is odd, then K_u and K_v are adjacent;
- (b) if p is even, then K_u and K_v are adjacent or $K_u \cap K_v \neq \emptyset$.

Proof. Let $k = \lfloor p/2 \rfloor$, and note that we must have $\ell \geq k$ as otherwise there would be a walk from u to v that goes through x and has length $2l < 2k \leq p$, which would contradict $d(u, v) = p$. We will prove (a) and (b) simultaneously by considering two possibilities for u and v .

We first consider the case in which u and v are in different components of $G_{>\ell-k}$. In this case it is clear that every path of length p joining u and v must contain a vertex from $G_{\ell-k}$ (and no vertices in $G_{<\ell-k}$). It is also easy to see that if p is even, then every path of length p joining u and v must have exactly one vertex in $G_{\ell-k}$. Since $d(u, v) = p$, this means that $K_u \cap K_v \neq \emptyset$. If p is odd, then every path of length p joining u and v must have exactly two vertices in $G_{\ell-k}$. This implies that K_u and K_v are adjacent.

We are now left to consider the case in which u and v are in the same connected component C of $G_{>\ell-k}$. Let $z \in G_{\ell-k+1}$ be an ancestor of u and let $z' \in G_{\ell-k+1}$ be an ancestor of v . Clearly, z and z' belong to C . We know by Lemma 4.2.1 that since G is chordal it is shadow complete, and so the neighbours of z and z' in $G_{\ell-k}$ form a clique. This means that either K_u and K_v are adjacent or $K_u \cap K_v \neq \emptyset$. However, if p is odd we cannot have $K_u \cap K_v \neq \emptyset$. \square

4.3 Adjacent-cliques graphs

In this section we prove Theorem 4.1.6. In order to prove this result we need to recall a specific characterisation of chordal graphs.

A *perfect elimination ordering* of a graph G is a linear ordering L of $V(G)$ such that, for every vertex $v \in V(G)$, the neighbours of v which are smaller than v in L form a clique. The following classical result is proved in [28, Section 7].

Proposition 4.3.1 (Fulkerson and Gross [28]).

A graph is chordal if and only if it has a perfect elimination ordering.

Proof of Theorem 4.1.6. By Proposition 4.3.1 we know G has a perfect elimination ordering. We fix one such ordering L . We say a vertex u is a *predecessor* of a vertex v if $uv \in E(G)$ and $u <_L v$. Moving along the ordering L , we colour the vertices of G in the following way. A vertex v gets a colour $a(v)$ which is different from $a(u)$ if u is a predecessor of v or u is a predecessor of a predecessor of v . Since the clique number of G is at most t and since L is a perfect elimination ordering, each vertex has at most $t - 1$ predecessors. Moreover, by choice of L we have that if v has $r \leq t - 1$ predecessors, the largest (with respect to L) of its predecessors has at most $t - r$ predecessors which are not already predecessors of v ; the second largest predecessor of v has at most $t - (r - 1)$ predecessors which are not already predecessors of v , and so on. Therefore, for any vertex v the set of predecessors and predecessors of a predecessor of v has size at most $r + (t - r) + (t - (r - 1)) + \cdots + (t - 1) \leq (t - 1) + 1 + 2 + \cdots + t - 1 = \binom{t+1}{2} - 1$. And so, the colouring a uses at most $\binom{t+1}{2}$ colours.

We define a colouring c on the vertices of $AC(G)$ in the following way. For every vertex z in $AC(G)$, with corresponding clique K in G , we set $\mu(K)$ as the smallest vertex of K with respect to L . Every vertex z is assigned the colour $c(z) = a(\mu(K))$. We claim that c is a proper colouring of $AC(G)$.

Let z and z^* be adjacent vertices in $AC(G)$. We must prove that the corresponding cliques in G , K and K^* , satisfy $a(\mu(K)) \neq a(\mu(K^*))$. Let $u, u' \in K$ and $v, v' \in K^*$ be vertices of G such that $u = \mu(K)$, $v = \mu(K^*)$ and $u'v' \in E(G)$. Without loss of generality we assume that $u' <_L v'$. If $v = v'$, we have that $u'v \in E(G)$. Otherwise, we have that both u' and v are predecessors of v' . Since L is a perfect elimination ordering, we also obtain $u'v \in E(G)$.

Note that a is a proper colouring of G . This means that if $u = u'$, we immediately get that $a(\mu(K)) = a(u) \neq a(v) = a(\mu(K^*))$ as desired. So assume $u \neq u'$. If $v <_L u'$, we have that both u and v are predecessors of u' , and so $uv \in E(G)$. This again gives us that $a(\mu(K)) \neq a(\mu(K^*))$. Otherwise, if $u' <_L v$ we have that u' is a predecessor of v . Since u is a predecessor of u' we also obtain that $a(\mu(K)) = a(u) \neq a(v) = a(\mu(K^*))$ by definition of a . \square

For a graph G , let $AIC(G)$ be a graph with the same vertex set as $AC(G)$ and an edge between two vertices z, z^* if and only if $zz^* \in E(AC(G))$ or the corresponding cliques K and K^* have vertices in common. For later use, we note a property of the colouring c we constructed in the previous proof.

Lemma 4.3.2.

Let G be a graph with clique number at most t . Colour the vertices of $AIC(G)$ with the colouring c , defined in the proof of Theorem 4.1.6. If two vertices z, z^ of $AIC(G)$ satisfy $zz^* \in E(AIC(G))$ and $c(z) = c(z^*)$, then we have that the corresponding cliques K, K^* satisfy $\mu(K) = \mu(K^*)$.*

Proof. We prove that if $\mu(K) \neq \mu(K^*)$, then $c(z) \neq c(z^*)$. By the proof of Theorem 4.1.6 we know that if K and K^* have no vertices in common, then z and z^* get different colours. If K and K^* have vertices in common and $\mu(K) \neq \mu(K^*)$, we will prove that $\mu(K)$ and $\mu(K^*)$ are adjacent in G . Since a is a proper colouring of G , this will tell us that $c(z) = a(\mu(K)) \neq a(\mu(K^*)) = c(z^*)$ which gives us the result.

If $\mu(K)$ and $\mu(K^*)$ are not adjacent in G , we have that neither of $\mu(K), \mu(K^*)$ belong to $K \cap K^*$. Therefore, the minimum vertex v in $K \cap K^*$ with respect to L is adjacent to $\mu(K)$ and $\mu(K^*)$, and $\mu(K), \mu(K^*)$ are smaller than v in L . But this contradicts the choice of L , since the neighbours of v which are smaller than v in L must form a clique, and so must be pairwise adjacent. \square

4.4 Exact distance graphs of chordal graphs

Theorem 4.1.5 will follow from the next lemma.

Lemma 4.4.1.

Let G be a connected chordal graph with clique number $t \geq 2$, let x be a vertex in G , and $p \geq 2$ an integer. For any non-negative integer ℓ , we have that

- (a) if p is odd, then there is a colouring h of $N^\ell(x)$ using at most $\binom{t}{2}$ colours such that if $u, v \in N^\ell(x)$ satisfy $uv \in E(G^{\lfloor p \rfloor})$, then $h(u) \neq h(v)$;
- (b) if p is even, then there is a colouring h' of $N^\ell(x)$ using at most $\binom{t}{2} \cdot \Delta(G)$ colours such that if $u, v \in N^\ell(x)$ satisfy $uv \in E(G^{\lfloor p \rfloor})$, then $h'(u) \neq h'(v)$.

Proof. Let $k = \lfloor p/2 \rfloor$ and note, just as in the proof of Lemma 4.2.3, that if $u, v \in N^\ell(x)$ satisfy $uv \in E(G^{\lfloor p \rfloor})$, then we must have $\ell \geq k$.

- (a) By Lemma 4.2.1 we know that $G_{\ell-k}$ is a chordal graph with clique number at most $t - 1$. By Theorem 4.1.6 we know that there is a proper colouring c of the vertices of $AC(G_{\ell-k})$ which uses at most $\binom{t}{2}$ colours.

For every vertex $y \in N^\ell(x)$ we consider the set of vertices $K_y \subseteq N^{\ell-k}(x)$ which are ancestors of y . By Corollary 4.2.2 we have that K_y induces a clique in $G_{\ell-k}$. Let z_y be the vertex corresponding to this clique in $AC(G_{\ell-k})$. Define the colouring h by assigning $h(y) = c(z_y)$ to every vertex $y \in N^\ell(x)$. Let $u, v \in N^\ell(x)$ be such that $uv \in E(G^{\lfloor p \rfloor})$. By Lemma 4.2.3 (a) we have that K_u and K_v are adjacent, which implies $z_u z_v \in E(AC(G_{\ell-k}))$. Therefore, we have $h(u) = c(z_u) \neq c(z_v) = h(v)$, as desired.

- (b) Recall that in the proof of Theorem 4.1.6 the colouring c is obtained by ordering the vertices of the original graph according to a perfect elimination ordering. Also, for each clique K in the graph, we consider the minimum vertex $\mu(K)$ in the clique with respect to the ordering.

We colour the vertices of $AIC(G_{\ell-k})$ with the colouring c . Additionally, for each vertex $w \in N^{\ell-k}(x)$ we choose an injective function $b_w : N(w) \rightarrow [\Delta(G)]$. For every vertex $y \in N^\ell(x)$ we choose an arbitrary vertex $\sigma(y)$ from $N^{k-1}(y) \cap N(\mu(K_y))$. The colouring h' assigns $h'(y) = (c(z_y), b_{\mu(K_y)}(\sigma(y)))$ to every vertex $y \in N^\ell(x)$. Clearly h' uses at most $\binom{t}{2} \cdot \Delta(G)$ colours.

Let $u, v \in N^\ell(x)$ be such that $uv \in E(G^{\lfloor p \rfloor})$. We must show that $h'(u) \neq h'(v)$. Suppose we have $K_u \cap K_v = \emptyset$. By Lemma 4.2.3 (b) we know that K_u and K_v are adjacent. As in part (a) we obtain $c(z_u) \neq c(z_v)$ and so $h'(u) \neq h'(v)$, as desired.

If we have $K_u \cap K_v \neq \emptyset$, we know that $z_u z_v \in E(AIC(G_{\ell-k}))$. Let us assume $c(z_u) = c(z_v)$, as otherwise we would have $h'(u) \neq h'(v)$. By Lemma 4.3.2 we obtain that the corresponding cliques K_u and K_v satisfy $\mu(K_u) = \mu(K_v)$. Now notice that $\sigma(u)$ must be different from $\sigma(v)$, as otherwise there would be a walk of length $p-2$ joining u and v and going through $\sigma(u)$, which would contradict $d(u, v) = p$. Since $b_{\mu(K_u)}$ is injective, we obtain $h'(u) \neq h'(v)$, as desired. \square

We now prove the general version of our main result.

Theorem (Theorem 4.1.5).

Let G be a chordal graph with clique number $t \geq 2$. Let p be a positive integer, $S = \{p_1, p_2, \dots, p_s\} \subseteq [p]$ and q be the number of even integers in S .

- (a) *If $1 \notin S$, then we have $\chi(G^{[\lfloor p_1 \rfloor]} \cup G^{[\lfloor p_2 \rfloor]} \cup \dots \cup G^{[\lfloor p_s \rfloor]}) \leq \binom{t}{2}^s \cdot \Delta(G)^q \cdot (p+1)$.*
- (b) *If $1 \in S$, then we have $\chi(G^{[\lfloor p_1 \rfloor]} \cup G^{[\lfloor p_2 \rfloor]} \cup \dots \cup G^{[\lfloor p_s \rfloor]}) \leq t \cdot \binom{t}{2}^{s-1} \cdot \Delta(G)^q \cdot (p+1)$.*

Proof. We may assume that G is connected. As we mentioned earlier, this theorem follows from Lemma 4.4.1. Here we prove (a) and leave (b) to the reader.

Fix a vertex $x \in V(G)$. Define a function $f : V(G) \rightarrow \{0, \dots, p\}$ which satisfies $f(u) = k$ for all $u \in N^\ell(x)$ with $\ell \equiv k \pmod{p+1}$. For each level $N^\ell(x)$ and integer $p_i \in \{p_1, p_2, \dots, p_s\}$, we define $g_{\ell,i}$ as the colouring of $N^\ell(x)$ guaranteed by Lemma 4.4.1, which assigns different colours to vertices of $N^\ell(x)$ having distance p_i . To each vertex $u \in N^\ell$ we assign a colour $F(u) = (f(u), g_{\ell,1}(u), g_{\ell,2}(u), \dots, g_{\ell,s}(u))$, and we do this for all ℓ . Notice that for every $1 \leq i \leq s$, each vertex $u \in N^\ell$ can only have distance p_i with vertices not in $N^{<\ell-p}(x) \cup N^{>\ell+p}(x)$. Hence, this colouring guarantees that, for all $1 \leq i \leq s$, u gets a different colour from v whenever u and v have distance p_i . \square

4.5 A remark on graphs with bounded genus

In this section we provide a way of obtaining from a connected graph H with genus $g \geq 0$ another connected graph G of genus g which has the additional property of being edge-maximal (with respect to having genus g), and which satisfies $\chi(H^{[\lfloor p \rfloor]}) \leq \chi(G^{[\lfloor p \rfloor]})$ for all positive p .

Proposition (Proposition 4.1.7).

Let H be a graph with genus $g \geq 0$. There is an edge-maximal graph G of genus g such that H is a subgraph of G and $\chi(H^{[lp]}) \leq \chi(G^{[lp]})$ for every positive integer p .

Proof. We may assume $V(H) \geq 3$, as otherwise the result is trivial. We may also assume that H is connected.

Fix an embedding of H in a surface of genus g . We first construct from H a graph H' of genus g having the property that all of its faces have a cycle as its boundary. This can be done without altering distances by means of the following operation. Suppose $y \in V(H)$ is a cut vertex. There is an ordering $x_1, x_2, \dots, x_{|N(y)|}$ of the vertices in $N(y)$ such that adding an edge between x_i and x_{i+1} (wherever such an edge does not already exist) would not create crossings. Using this ordering we add a path of length 2 between x_i and x_{i+1} (modulo $|N(y)|$) if there is no edge joining the pair. Clearly y ceases to be a cut vertex after this operation, and no new cut vertices are created. We repeat this operation until there are no cut vertices. It is easy to see that H' satisfies that all of its faces have a cycle as its boundary, and that for every $u, v \in V(H)$ we have $d_H(u, v) = d_{H'}(u, v)$.

If H' is not an edge-maximal graph of genus g , then there is a face of H having as its boundary a cycle C_k , with vertices z_0, \dots, z_{k-1} , for some $k > 3$. Inside this face we draw a cycle C_{k-1} with edges e_1, \dots, e_{k-1} . For all $1 \leq i \leq k-1$, we add edges joining z_i with the endvertices of e_i . We also add an edge joining z_0 with the common endvertex of e_1 and e_{k-1} . It is easy to see that this can be done in such a way that no crossings are made, the area between C_k and C_{k-1} is triangulated and C_{k-1} is the boundary of a face. Figure 4.1 shows how to do this for $k = 7$. We call the resulting embedded graph F . If F is not a triangulation, we repeat the operation on F , and we do this until we get a maximal graph G of genus g .

To prove that $\chi(G^{[lp]}) \geq \chi(H^{[lp]})$ for every positive integer p , it is enough to prove that F satisfies $\chi(F^{[lp]}) \geq \chi(H^{[lp]})$ for every positive p . This can be done by showing that every pair of vertices $u, v \in V(H)$ satisfies $d_F(u, v) = d_H(u, v)$. Note that if we had $d_F(u, v) < d_H(u, v)$, it would imply that there is a uv -path in F of length at most $d_H(u, v) - 1$ containing vertices of C_{k-1} . In that case, we obtain that there is a pair of vertices $x, y \in V(C_k)$ with $d_F(x, y) < d_H(x, y)$, such that there is in F an xy -path of length at most $d_H(x, y) - 1$ with vertices in C_{k-1} and no vertices outside of $C_k \cup C_{k-1}$. Hence, it is sufficient to show that every pair

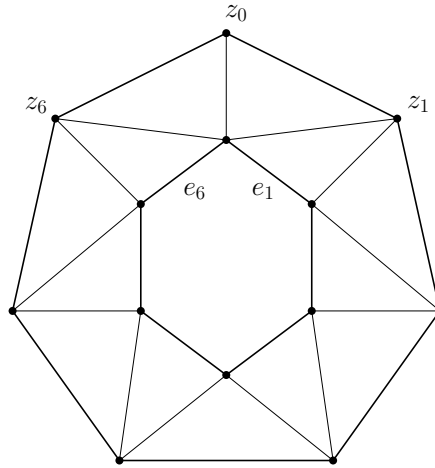


Figure 4.1: Drawing a C_6 inside a face of H which has a C_7 as its boundary.

of vertices $x, y \in V(C_k)$ satisfies $d_{F[C_k \cup C_{k-1}]}(x, y) = d_{C_k}(x, y)$. By the construction inside the face having C_k as its boundary, this is easy to check. \square

Chapter 5

When are sparse graph classes closed under addition?

5.1 Introduction

5.1.1 Adding edges, graphs, and classes

In order to prove that a class of graphs has a certain property, it is common to first prove that all edge-maximal graphs in that class have that property. In Chapter 2, for example, we found upper bounds for the generalised colouring numbers of maximal planar graphs and, since these numbers cannot increase when edges are removed, we immediately obtained upper bounds for all planar graphs.

One of the reasons why maximality is a desirable property is because maximal graphs of a given class have stronger restrictions on their structure, which tends to translate into useful structural properties. For example, maximal planar graphs are triangulated, meaning that they have an embedding in the plane in which every face is bounded by exactly three edges, a most useful property. But, as this classical example suggests, in many cases, maximality is a *means* to obtain a property and not desirable *for its own sake*.

If we really wish to obtain a particular property by means of adding edges, we may allow ourselves to go beyond maximality. We may add edges to each graph of a class until all of them have the desired property, and try to guarantee that the resulting class does not have

properties which amply differ from those of the original class. Let us see an example of this. A graph G is said to be k -outerplanar for $k = 1$ if G is outerplanar, and for $k > 1$ if G has a planar embedding such that if all vertices on the outer face are deleted, the remaining graph is $(k - 1)$ -outerplanar. Biedl [9] proved that not all k -outerplanar graphs can be triangulated so that the result is k -outerplanar, but all can be triangulated so that the result is $(k + 1)$ -outerplanar.

The notion of classes with bounded expansion provides a framework in which many different graph classes can be studied under the same light. At least 12 characterizations of classes with bounded expansion can be found in the literature [62, 65, 69, 84]. In this chapter we will study some circumstances in which we can add edges to all graphs in a class with bounded expansion, guaranteeing that all of them get a certain property, while obtaining a class with bounded expansion as a result. The robustness of the notion of classes with bounded expansion will allow us to require strong properties and yet arrive at a class which, in many ways, does not differ greatly from the original one.

Recall that a *Hamiltonian path* in a graph G is a path that visits each vertex of $V(G)$. One of our main objectives will be to add edges to the graphs in a fixed class so as to guarantee that each gets a Hamiltonian path, while preserving the structural properties of the class as much as possible. Let us first see that this needs to be done in a careful way, even if we start with the class of trees. For an odd integer $n \geq 3$, consider the star forest F_n with n stars A_1, \dots, A_n , each with $n - 2$ leaves. We can add a Hamiltonian path in the following way: let consecutive vertices belong to different stars and, for each i , every time the path visits A_i it jumps to a star it has not visited from A_i before. If in the resulting graph F'_n we contract each star, we get a clique of size n . Since stars have radius 1, we have that $\nabla_1(F'_n) = (n - 2)/2$. Therefore, the class of graphs obtained from F_3, F_5, \dots by adding, to each of them, a Hamiltonian path in this way does not have bounded expansion.

In contrast, we show that there is a way of adding edges to a graph of tree-width t in such a way that the resulting graph has a Hamiltonian path and tree-width at most $t + 2$. We prove the following theorem in Section 5.2.

Theorem 5.1.1.

Let G be a graph with tree-width at most t . There exist a graph H , with $|V(G)| = |V(H)|$, such that H has tree-width at most $t + 2$, has a Hamiltonian path and is a supergraph of G .

Notice that adding edges to a graph G in order to obtain a Hamiltonian path can be seen as “adding” two graphs, G and a path of length $|V(G)|$. Let us formalise this intuition.

Definition 5.1.2. We say that two graphs G_1 and G_2 *can be summed* to another graph H , if there is a bijection $h: V(G_1) \rightarrow V(G_2)$ such that by identifying v and $h(v)$ (and possibly deleting multiple edges) we obtain the graph H .

We extend this notion to graph classes as follows.

Definition 5.1.3. Two classes of graphs \mathcal{C}_1 and \mathcal{C}_2 *can be summed* to another class \mathcal{K} , if for every integer $n \geq 1$ and every pair of graphs $G_1 \in \mathcal{C}_1$, $G_2 \in \mathcal{C}_2$, with $n = |V(G_1)| = |V(G_2)|$, there is a graph $H \in \mathcal{K}$ such that G_1 and G_2 can be summed to H .

Notice that, by the freedom given by the choice of the bijections, the word “can” is fundamental here. Our construction above tells us that the class of all forests and the class of all paths can be summed to a class that does not have bounded expansion. Meanwhile, Theorem 5.1.1 tells us that these same two classes can be summed to a class of tree-width 2. In fact, Theorem 5.1.1 easily implies the following corollary.

Corollary 5.1.4.

Let \mathcal{C}_1 be a class of graphs with tree-width at most t , and let \mathcal{C}_2 be a class containing only paths. Then \mathcal{C}_1 and \mathcal{C}_2 can be summed to a class of graphs with tree-width at most $t + 1$.

Since adding an edge to a graph can only increase the tree-width of the graph by one, we also get that any class of graphs with tree-width at most t , and the class of all cycles can be summed to a class of graphs with tree-width at most $t + 2$.

Let us say that a class with property P_1 and a class with property P_2 *can be summed* to a class with property P_3 , if for every pair \mathcal{C}_1 , \mathcal{C}_2 of classes with properties P_1 and P_2 , respectively, there exists some class \mathcal{K} with property P_3 such that \mathcal{C}_1 and \mathcal{C}_2 can be summed to \mathcal{K} .

We do not know whether a minor closed class and the class of all paths can be summed to a (proper) minor closed class. But we can prove that a minor closed class and the class

of all paths can be summed to a class of bounded expansion. Moreover, by drawing upon results related to the concept of *page number* of a graph, without much effort, we prove the following (perhaps surprising) result in Section 5.3.

Theorem 5.1.5.

Any pair of (proper) minor closed classes can be summed to a class with bounded expansion.

We do not come close to replicating this result for classes closed under taking topological minors. For the moment, we do not know whether a class closed under taking topological minors and the class of all paths can be summed to a class with bounded expansion. Nevertheless, we prove that we can add edges to each graph of a class which excludes a fixed complete topological minor so as to guarantee that each graph gets a spanning tree of degree at most 3 and that the resulting class has bounded expansion.

Theorem 5.1.6.

Let t be a positive integer and let \mathcal{C} be the class of graphs excluding K_t as a topological minor. There exists a class \mathcal{K} with bounded expansion such that the following holds. For every $G \in \mathcal{C}$ there is a graph $H \in \mathcal{K}$ and a tree T of degree at most 3, with $|V(G)| = |V(T)|$, such that G and T can be summed to H .

In Section 5.4 we prove the following theorem, which together with a previous result of Kreutzer, Pilipczuk, Rabinovich, and Siebertz [53], implies Theorem 5.1.6. For a graph G denote by $\binom{V(G)}{2}$ the set of all unordered pairs of vertices of G . For $F \subseteq \binom{V(G)}{2}$ we denote by $G + F$ the graph obtained from G by adding to G all edges in $F \setminus E(G)$.

Theorem 5.1.7.

For every graph G and every ordering L of $V(G)$, there exist a function $f: \mathbb{N} \rightarrow \mathbb{N}$ and a set of unordered pairs $F \subseteq \binom{V(G)}{2}$ such that the graph $T = (V(G), F)$ is a tree of maximum degree at most 3 and, for all k , we have

$$\max_{y \in V(G)} S_k[G + F, L, y] \leq f\left(\max_{y \in V(G)} |S_{2k}[G, L, y]|\right).$$

As we shall see, this result is a major step towards proving that Theorem 5.1.6 holds when we allow \mathcal{C} to be any class with bounded expansion. In order to state Theorem 5.1.7

more precisely (exposing the function f), and see how it implies Theorem 5.1.6, we need to introduce another generalisation of the colouring number: k -admissibility.

5.1.2 k -admissibility and uniform orderings

Let G be a graph, L a linear ordering of $V(G)$, and k a positive integer. The k -admissibility $\text{adm}_k[G, L, y]$ of a vertex y with respect to L is defined as the maximum size of a family \mathcal{P} of paths that satisfies the following two properties:

- each path in \mathcal{P} has length at most k , starts in y , ends in a vertex that is smaller than y in L , and all its internal vertices are larger than y in L ;
- the paths in \mathcal{P} are pairwise vertex-disjoint, apart from sharing the start vertex y .

The k -admissibility $\text{adm}_k(G)$ of a graph G is defined as follows:

$$\text{adm}_k(G) = \min_L \max_{y \in V(G)} \text{adm}_k[G, L, y] + 1.$$

It is clear that for every graph G we have $\text{adm}_k(G) \leq \text{scol}_k(G) \leq \text{wcol}_k(G)$. Moreover, Dvořák [21] (who introduced the notion of admissibility) proved that

$$\text{scol}_k(G) \leq (\text{adm}_k(G))^k. \quad (5.1)$$

Note that unlike $S_k[G, L, y]$ and $W_k[G, L, y]$, which are sets, $\text{adm}_k[G, L, y]$ is a number.

Using these relationships and the fact that $\text{wcol}_k(G) \leq (\text{scol}_k(G))^k$, we get the following corollary of Zhu's characterization of classes with bounded expansion (Theorem 1.2.3).

Corollary 5.1.8.

A class of graphs \mathcal{K} has bounded expansion if and only if there exist constants c_k , $k = 1, 2, \dots$ such that $\text{adm}_k(G) \leq c_k$ for all k and all $G \in \mathcal{K}$.

This corollary and the following result of Kreutzer *et al.* [53] tell us that there is a way of adding edges to each graph of a class excluding a complete topological minor so as to guarantee that each gets a spanning tree of bounded maximum degree and the resulting class has bounded expansion.

Theorem 5.1.9 (Kreutzer *et al.* [53]).

Let t be a positive integer. There is a constant $\Delta(t)$ and a function $f_t: \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds. For every graph G which excludes K_t as a topological minor, there is a supergraph H of G , with $|V(H)| = |V(G)|$, such that $\text{adm}_k(H) \leq f_t(k)$ for all $k \in \mathbb{N}$ and such that H contains a spanning tree with maximum degree at most $\Delta(t)$.

The notion of *uniform orderings* is key in the proof that Kreutzer *et al.* gave of this result. We say that a class \mathcal{C} admits *uniform orderings* if there is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for each $G \in \mathcal{C}$ there is a linear order of $V(G)$ such that $\max_{y \in V(G)} \text{adm}_k[G, L, y] \leq f(k)$ for all k . The flat decompositions that we developed in Chapter 2 implicitly tell us that any class excluding a fixed complete minor admits uniform orderings. Using this together with the decomposition theorem for graphs with excluded topological minors, due to Grohe and Marx [32], Kreutzer *et al.* [53] proved the following.

Theorem 5.1.10 (Kreutzer *et al.* [53]).

Let t be a positive integer. If G excludes K_t as a topological minor, then there exists a constant $\gamma(t)$ and a uniform order L such that $\max_{y \in V(G)} \text{adm}_k[G, L, y] \leq \gamma \cdot k$ for all positive integers k .

One of the main contributions of this chapter is the following theorem.

Theorem 5.1.11.

For every graph G and every ordering L of $V(G)$, there exists a set of unordered pairs $F \subseteq \binom{V(G)}{2}$ such that the graph $T = (V(G), F)$ is a tree of maximum degree at most 3 and, for all k , we have

$$\max_{y \in V(G)} \text{adm}_k[G + F, L, y] \leq 2 + 3 \cdot \max_{y \in V(G)} |S_{2k}[G, L, y]|.$$

It is easy to see that this result, together with Theorem 5.1.10, gives us Theorem 5.1.6. Let \mathcal{C} be a class of graph excluding K_t as a minor. By Theorem 5.1.10, each graph $G \in \mathcal{C}$ has a uniform ordering L_G . For each graph $G \in \mathcal{C}$ and corresponding ordering L_G , we pick a set F_G of unordered pairs as that guaranteed by Theorem 5.1.11. The linear upper bound on $\max_{y \in V(G)} \text{adm}_k[G, L_G, y]$ guarantees, through inequality (5.1), that there is an exponential function g which bounds $\max_{y \in V(G)} |S_{2k}[G, L_G, y]|$. By the inequality of Theorem 5.1.11,

the graph class formed by applying this algorithm to all graphs in \mathcal{C} has bounded expansion with corresponding function g . Moreover, this new class admits uniform orderings with corresponding function g .

Theorem 5.1.6 improves Theorem 5.1.9 in two ways. Theorem 5.1.6 makes the degree of the spanning tree independent of the initial class and very close to forcing a Hamiltonian path. Additionally, the tool we develop, Theorem 5.1.11, paves the way for proving a version of Theorem 5.1.6 where \mathcal{C} can be any class with bounded expansion. What would be needed is proving that any class with bounded expansion admits uniform orderings. We leave as an open problem whether a class closed under taking topological minors (or, indeed, any class with bounded expansion) plus the class of all paths can be summed to a class with bounded expansion.

It must be noted that, although we have given here a purely Graph Theoretical motivation, Theorem 5.1.11 and similar results, like Theorem 5.1.9, find their original motivation in Model Theory. Towards the end of the chapter we will study this original motivation, and sketch how Theorem 5.1.11 implies a new result from the theory of Parametrised Complexity.

The rest of the chapter is organised as follows. In Section 5.2, we prove Theorem 5.1.1. In Section 5.3, we introduce the notion of *page number* and the results needed to prove Theorem 5.1.5. In Section 5.4, we prove Theorem 5.1.11. We conclude with a study of first-order logic, leading to an application of Theorem 5.1.11.

5.2 Classes with bounded tree-width plus the class of paths

In this section we prove Theorem 5.1.1, which implies that a class of graphs with tree-width at most t and the class of all paths can be summed to a class with tree-width at most $t + 2$.

Before dealing with the proof, let us recall some definitions already studied in Chapters 2 and 4.

Rooting a tree T in some vertex $w \in V(T)$ imposes *child-parent* and *ancestor-descendant* relations in T . More precisely, a vertex v is an *ancestor* of a vertex $u \neq v$, if it lies on the unique path from u to the root w , and it is the *parent* of u if it is the neighbour of u on this path.

Recall that the bags of a tree-decomposition $(T, (X_y)_{y \in V(T)})$ of a graph G must satisfy:

- (1) $\bigcup_{y \in V(T)} X_y = V(G)$;
- (2) for every edge $uv \in E(G)$, there is a $y \in V(T)$ such that $u, v \in X_y$; and
- (3) if $v \in X_y \cap X_{y'}$ for some $y, y' \in V(T)$, then $v \in X_{y''}$ for all y'' that lie on the unique path between y and y' in T .

Recall also that every graph of tree-width t has a smooth tree-decomposition of width t , that is, a tree-decomposition where all bags have $t + 1$ vertices and adjacent bags share exactly t vertices.

Proof of Theorem 5.1.1. It is easy to see that we can assume that G is connected. Say that for every connected component G_1, G_2, \dots, G_p of G , we find corresponding graphs H_1, H_2, \dots, H_p with Hamiltonian paths P_1, P_2, \dots, P_p , and corresponding endpoints $v_1, v'_1, \dots, v_p, v'_p$. The path $v_1 P_1 v'_1 v_2 P_2 v'_2 v_3 \dots v_{p-1} P_{p-1} v'_{p-1} v_p P_p v'_p$ is Hamiltonian in G , and adding those extra edges does not increase the tree-width (unless each component was a single vertex, in which case, adding a Hamiltonian path to each component does not increase the tree-width).

Let $(T, (X_y)_{y \in V(T)})$ be a smooth tree-decomposition of width t of G . We can assume that no two nodes of T correspond to the same bag. We will construct another smooth tree-decomposition $(T^*, (X_y)_{y \in V(T^*)})$ of G in the following way. Let a be a node of T , and assign it as the root of T . For every node b such that $|X_a \cap X_b| = t$ and $ab \notin E(T)$, we delete the edge joining b with its parent, and add the edge ab . We call the resulting graph $T(0)$, and also root it at a . It is easy to see that $T(0)$ and the bags of the original decomposition also form a tree-decomposition of G . Let b, c be children of a in $T(0)$, and let b' and c' be nodes belonging, respectively, to the subtrees formed by b and its descendants, and c and its descendants. By construction, and by property (3) of the bags of a tree-decomposition, we know that unless $b' = b$ and $c' = c$, we cannot have $|X_{b'} \cap X_{c'}| = t$. We repeat the procedure of edge deletion and addition on each subtree of $T(0)$ formed by a child of a and its descendants. We call the resulting tree $T(1)$, and note that, together with the bags of the original decomposition, it also forms a tree-decomposition of G . We repeat the procedure inductively on level i of $T(i)$, and call the resulting tree T^* . Note that, when constructing level i , we do not consider the nodes in levels smaller than i , and so the procedure eventually stops. For now, we retain the original bags.

We will construct H by adding appropriate edges to G . Our procedure will be guided by a depth-first search on T^* , rooted at a . For $1 \leq i \leq |V(T)| - 1$, let a_i be the i -th node to be visited by this depth-first search, and let $a_0 = a$. Notice that for any bag X we can add edges to $G[X]$ without affecting the width of the decomposition. In particular, we can add edges to $G[X_{a_0}]$ so as to guarantee it has a Hamiltonian path with the endpoints of our choice. Where they are not already in G , we add edges to $G[X_{a_0}]$ so as to create a Hamiltonian path with an end in u , for some $u \in X_{a_0} \cap X_{a_1}$. Since u belongs to X_{a_1} , the width of the decomposition will not be affected if we also add the edge uv , for the unique $v \in X_{a_1} \setminus X_{a_0}$.

For $1 \leq i \leq |V(T^*)| - 2$ let b_i be the parent of a_i in T^* , and denote by $v(a_i)$ the unique vertex in $X_{a_i} \setminus X_{b_i}$. If $a_i a_{i+1}$ is an edge of T^* , then $|X_{a_{i+1}} \setminus X_{b_i}| = 2$. This implies that $v(a_i)$ belongs to $X_{a_{i+1}}$. Therefore, if it is not already in G , we can add the edge $v(a_i)v(a_{i+1})$ without affecting the width of the decomposition. If after doing this for every $1 \leq i \leq |V(T^*)| - 2$ such that $a_i a_{i+1} \in E(T^*)$ we have already created a Hamilton path, we let H be the resulting graph. In this case, note that T^* together with the original bags form a tree-decomposition of H of width t . Otherwise, we need to add edges of the form $v(a_i)v(a_{i+1})$, for $1 \leq i \leq |V(T^*)| - 2$, such that $a_i a_{i+1} \notin E(T^*)$. In the remaining part of the proof we show how to modify T^* and the bags, so as to create a tree-decomposition of width at most $t + 2$ for the graph H which results from adding these edges.

For every $1 \leq i \leq |V(T^*)| - 2$ such that $a_i a_{i+1} \notin E(T^*)$, we proceed as follows. Let a_j , $1 \leq j \leq i$, be the child of b_{i+1} which was visited last before a_{i+1} by the depth-first search of T^* . We delete the edge $b_{i+1} a_{i+1}$ from $E(T^*)$ and add the edge $a_j a_{i+1}$. Notice that $|X_{a_{i+1}} \setminus X_{a_j}| \geq 2$, and that necessarily $v(a_{i+1}) \in X_{a_{i+1}} \setminus X_{a_j}$. To all the bags in the path $a_j a_{j+1} \dots a_i$ we add the vertex $v(a_{i+1})$. If after this operation $|X_{a_{i+1}} \setminus X_{a_j}| > 0$ we add to X_{a_j} the unique vertex in $X_{a_{i+1}} \setminus X_{a_j}$. It is not hard to see that, after updating the bags as above, every path created in T^* by adding the edge $a_j a_{i+1}$ satisfies property (3). Moreover, since we have added $v(a_{i+1})$ to X_{a_i} , we can add the edge $v(a_i)v(a_{i+1})$ without violating property (2). Therefore, the new decomposition satisfies the three properties of the bags of a tree-decomposition. All we need to check now is that the width of the new decomposition is $t + 2$. But this follows from the fact that each path $a_j a_{j+1} \dots a_i$ is only updated once, hence every bag gets vertices added to it at most once, and at most two of these. \square

5.3 Page number and adding minor closed classes

In this section we prove Theorem 5.1.5 which tells us that any two minor closed classes can be summed to a class with bounded expansion. The proof of this result is easy in light of some known results related to the *page number* of a graph.

For a graph G and a linear ordering L of $V(G)$, we say that two edges $e_1 = x_1y_1$ and $e_2 = x_2y_2$ are *crossing* if $x_1 <_L x_2 <_L y_1 <_L y_2$. If a set $B \subseteq E(G)$ has no crossing edges in L , we say that B is a *page* of L . The *page number* of a graph G is the minimum k for which there is a linear ordering L of $V(G)$ such that there is a partition of $E(G)$ into k pages B_1, B_2, \dots, B_k of L .

Using the structural theory of Robertson and Seymour, Blankenship [10] proved that minor closed classes have bounded page number.

Theorem 5.3.1 (Blankenship [10]).

For every (proper) minor closed class there is a constant k such that every graph in the class has page number at most k .

Additionally, Nešetřil *et al.* [65] proved the following.

Theorem 5.3.2 (Nešetřil, Ossona de Mendez, and Wood [65]).

Let k be a positive integer and let \mathcal{K} be a class of graphs with page number at most k . Then \mathcal{K} has bounded expansion.

We now prove Theorem 5.1.5 by combining these two theorems.

Proof of Theorem 5.1.5. Let \mathcal{K}_1 and \mathcal{K}_2 be (proper) minor closed classes. By Theorem 5.3.1 we know that there are constants k_1 and k_2 such that, for $1 \leq i \leq 2$, every graph in \mathcal{K}_i has page number at most k_i . For an arbitrary integer $n \geq 1$, let $G \in \mathcal{K}_1$ and $H \in \mathcal{K}_2$, be a pair of graphs such that $|V(G)| = |V(H)| = n$. Let L_G and L_H be orderings of $V(G)$ and $V(H)$, respectively, such that they witness the page number of the corresponding graph. For every $1 \leq i \leq |V(G)|$, we identify the i -th vertex of L_G with the i -th vertex of L_H . By definition of the page number and our choice of L_G and L_H , we know that $E(G)$ can be partitioned into $c_1 \leq k_1$ pages of L_G and that $E(H)$ can be partitioned into $c_2 \leq k_2$ pages of $L_H = L_G$. Therefore, the edges of the resulting graph can be partitioned into $c_1 + c_2 \leq k_1 + k_2$ pages

of L_G (even if we allow multiple edges). By our choice of G, H and n , we conclude that \mathcal{K}_1 and \mathcal{K}_2 can be summed to a class of graphs with page number at most $k_1 + k_2$. By Theorem 5.3.2 any such class has bounded expansion. \square

5.4 Adding spanning trees of small degree while preserving k -admissibility

In this section we prove Theorem 5.1.11. The main step towards this goal is the corresponding statement for connected graphs, as expressed in the following lemma.

Lemma 5.4.1.

For every connected graph G and every ordering L of $V(G)$, there exists a set of unordered pairs $F \subseteq \binom{V(G)}{2}$ such that the graph $T = (V(G), F)$ is a tree of maximum degree at most 3 and, for all k , we have

$$\max_{y \in V(G)} \text{adm}_k[G + F, L, y] \leq 3 \cdot \max_{y \in V(G)} |S_{2k}[G, L, y]|.$$

We first show that Theorem 5.1.11 follows easily from Lemma 5.4.1.

Proof of Theorem 5.1.11, assuming Lemma 5.4.1. Let G be a (possibly disconnected) graph, and let G_1, \dots, G_p be the connected components of G . For each $1 \leq i \leq p$, let L_i be the ordering obtained by restricting L to $V(G_i)$. It is clear that for all k we have $\max_{y \in V(G_i)} |S_{2k}[G_i, L_i, y]| \leq \max_{y \in V(G)} |S_{2k}[G, L, y]|$.

For each $1 \leq i \leq p$, apply Lemma 5.4.1 to G_i and L_i , obtaining a set of unordered pairs F_i such that $T_i = (V(G_i), F_i)$ is a tree of maximum degree at most 3 and, for all k , we have

$$\max_{y \in V(G_i)} \text{adm}_k[G_i + F_i, L_i, y] \leq 3 \cdot \max_{y \in V(G_i)} |S_{2k}[G_i, L_i, y]| \leq 3 \cdot \max_{y \in V(G)} |S_{2k}[G, L, y]|.$$

Also for each $1 \leq i \leq p$, select a vertex v_i of G_i with degree at most 1 in T_i ; since T_i is a tree, such a vertex exists. Define

$$F = \{v_1 v_2, v_2 v_3, \dots, v_{p-1} v_p\} \cup \bigcup_{i=1}^p F_i.$$

It is clear that $T = (V(G), F)$ is a tree. Observe that it has maximum degree at most 3. This is because each vertex v_i had degree at most 1 in its corresponding tree T_i , and hence its degree can grow to at most 3 after adding edges $v_{i-1}v_i$ and v_iv_{i+1} .

It remains to show that $\max_{y \in V(G)} \text{adm}_k[G + F, L, y] \leq 2 + 3 \cdot \max_{y \in V(G)} |S_{2k}[G, L, y]|$ for all k . Take any k and any vertex u of G , say $u \in V(G_i)$, and let \mathcal{P} be a set of paths of length at most k that start in u , are pairwise vertex-disjoint (apart from u), and end in vertices smaller than u in L while internally traversing only vertices larger than u in L . Notice that at most two of the paths from \mathcal{P} can use any of the edges from the set $\{v_1v_2, v_2v_3, \dots, v_{p-1}v_p\}$, since any such path has to use either $v_{i-1}v_i$ or v_iv_{i+1} . The remaining paths are entirely contained in $G_i + F_i$, and hence their number is bounded by $\max_{y \in V(G_i)} \text{adm}_k[G_i + F_i, L_i, y] \leq 3 \cdot \max_{y \in V(G)} |S_{2k}[G, L, y]|$. The theorem follows. \square

In the remainder of this section we focus on Lemma 5.4.1.

Proof of Lemma 5.4.1. We begin our proof by showing how to construct the set F . This will be a two step process, starting with an *elimination tree*. For a connected graph G and an ordering L of $V(G)$, we define the (rooted) *elimination tree* $Q(G, L)$ of G imposed by L (see [11, 74]) as follows. If $V(G) = \{v\}$, then the rooted elimination tree $Q(G, L)$ is just the tree on the single vertex v . Otherwise, the root of $Q(G, L)$ is the vertex w that is the smallest with respect to the ordering L in G . For each connected component C of $G - w$ we construct a rooted elimination tree $Q(C, L|_{V(C)})$, where $L|_{V(C)}$ denotes the restriction of L to the vertex set of C . These rooted elimination trees are attached below w as subtrees by making their roots into children of w . Thus, the vertex set of the elimination tree $Q(G, L)$ is always equal to the vertex set of G . See Figure 5.1 for an illustration. The solid black lines are the edges of G ; the dashed blue lines are the edges of $Q = Q(G, L)$. The ordering L is given by the numbers written in the vertices.

Given an ordering L of G , let $Q = Q(G, L)$ be the rooted elimination tree of G imposed by L . For a vertex u , by G_u we denote the subgraph of G induced by all descendants of u in Q , including u . The following properties follow easily from the construction of a rooted elimination tree.

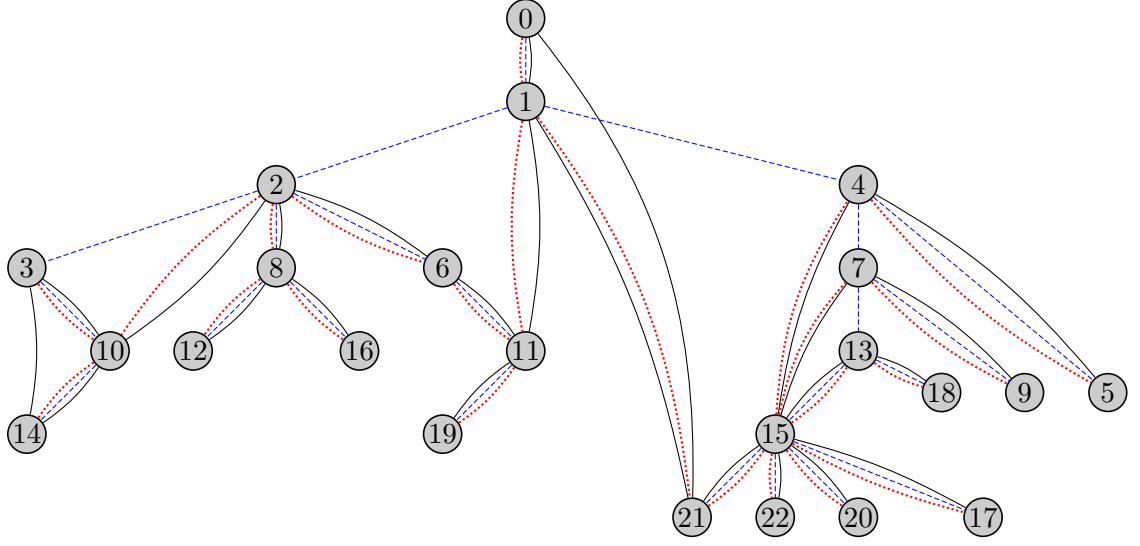


Figure 5.1: A graph G (solid black lines), the elimination tree Q (dashed blue lines), and the tree U (dotted red lines). Numbering of nodes reflects the ordering L .

Claim 5.4.2. *The following assertions hold.*

- (a) *For each $u \in V(G)$, the subgraph G_u is connected.*
- (b) *Whenever a vertex u is an ancestor of a vertex v in Q , we have $u <_L v$.*
- (c) *For each $uv \in E(G)$ with $u <_L v$, u is an ancestor of v in Q .*
- (d) *For each $u \in V(G)$ and each child v of u in Q , u has at least one neighbour in $V(G_v)$.*

Proof of claim. Assertions (a) and (b) follow immediately from the construction of Q . For assertion (c), suppose that u and v are not bound by the ancestor-descendant relation in Q , and let w be their lowest common ancestor in Q . Then u and v would be in different connected components of $G_w - w$, hence uv could not be an edge; a contradiction. It follows that u and v are bound by the ancestor-descendant relation, implying that u is an ancestor of v , due to $u <_L v$ and assertion (b). Finally, for assertion (d), recall that by assertion (a) we have that G_u is connected, whereas by construction G_v is one of the connected components of $G_u - u$. Hence, in G there is no edge between $V(G_v)$ and any of the other connected components of $G_u - u$. If there was no edge between $V(G_v)$ and u as well, then there would be no edge between $V(G_v)$ and $V(G_u) \setminus V(G_v)$, contradicting the connectivity of G_u . \lrcorner

We now choose a set of edges $B \subseteq E(G)$ as follows. For every vertex u of G and every child v of u in Q , select an arbitrary neighbour $w_{u,v}$ of u in G_v ; such a neighbour exists by

Claim 5.4.2 (d). Then let B_u be the set of all edges $uw_{u,v}$, for v ranging over the children of u in Q . Define

$$B = \bigcup_{u \in V(G)} B_u.$$

Let U be the graph spanned by all the edges in B , that is, $U = (V(G), B)$. In Figure 5.1, the edges of U are represented by the dotted red lines.

Claim 5.4.3. *The graph U is a tree.*

Proof of claim. Notice that for each $u \in V(G)$, the number of edges in B_u is equal to the number of children of u in Q . Since every vertex of G has exactly one parent in Q , apart from the root of Q , we infer that

$$|B| \leq \sum_{u \in V(G)} |B_u| = |V(G)| - 1.$$

Therefore, since B is the edge set of U , to prove that U is a tree it suffices to prove that U is connected. To this end, we prove by a bottom-up induction on Q that for each $u \in V(G)$, the subgraph $U_u = (V(G_u), B \cap (V(G_u)))$ is connected. Note that for the root w of Q this claim is equivalent to $U_w = U$ being connected.

Take any $u \in V(G)$, and suppose by induction that for each child v of u in Q , the subgraph U_v is connected. Observe that U_u can be constructed by taking the vertex u and, for each child v of u in Q , adding the connected subgraph U_v and connecting it to u via edge $uw_{u,v} \in B_u$. Thus, U_u constructed in this manner is also connected, as claimed. \lrcorner

By Claim 5.4.3 we have that U is a spanning tree of G , however its maximum degree may be (too) large. The idea is to use U to construct a new tree T with maximum degree at most 3 (on the same vertex set $V(G)$). The way we constructed U will enable us to argue that adding the edges of T to the graph G does not change the admissibility too much.

Give U the same root as the elimination tree Q . From now on we treat U as a rooted tree, which imposes parent-child and ancestor-descendant relations in U as well. Note that the parent-child and ancestor-descendant relations in Q and in U may be completely different. For instance, consider vertices 4 and 15 in the example from Figure 5.1: the vertex 4 is a child of 15 in U , and an ancestor of 15 in Q .

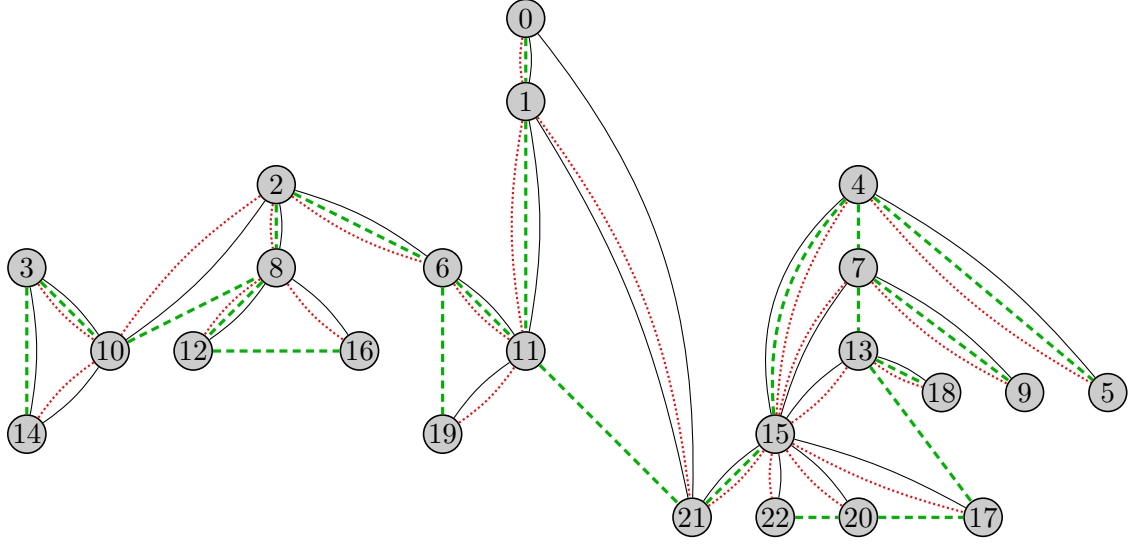


Figure 5.2: A graph G (solid black lines), the tree U (dotted red lines), and the tree T (thick dashed green lines).

For every $u \in V(G)$, let (x_1, \dots, x_p) be an enumeration of the children of u in U , such that $x_i <_L x_j$ if $i < j$. Let $F_u = \{ux_1, x_1x_2, x_2x_3, \dots, x_{p-1}x_p\}$, and define

$$F = \bigcup_{u \in V(G)} F_u \quad \text{and} \quad T = (V(G), F).$$

See Figure 5.2 for an illustration.

Claim 5.4.4. *The graph T is a tree with maximum degree at most 3.*

Proof of claim. Notice that for each $u \in V(G)$, we have that $|F_u|$ is equal to the number of children of u in U . Every vertex of G apart from the root of U has exactly one parent in U , hence

$$|F| \leq \sum_{u \in V(G)} |F_u| = |V(G)| - 1.$$

Therefore, to prove that T is a tree, it suffices to argue that it is connected. This, however, follows immediately from the fact that U is connected, since for each edge in U there is a path in T that connects the same pair of vertices.

Finally, it is easy to see that each vertex u is incident to at most 3 edges of F : at most one leading to a child of u in U , and at most 2 belonging to F_v , where v is the parent of u

in U . ┘

It remains to check that adding F to G does not change the admissibility too much.

Take any vertex $u \in V(G)$ and examine its children in U . We partition them as follows. Let Z_u^\uparrow be the set of those children of u in U that are its ancestors in Q , and let Z_u^\downarrow be the set of those children of u in U that are its descendants in Q . By the construction of U and by Claim 5.4.2(c), each child of u in U is either its ancestor or descendant in Q . By Claim 5.4.2(a), this is equivalent to saying that Z_u^\uparrow , respectively Z_u^\downarrow , comprise the children of u in U that are smaller, respectively larger, than u in L . Note that by the construction of U , the vertices of Z_u^\downarrow lie in pairwise different subtrees rooted at the children of u in Q , thus u is the lowest common ancestor in Q of every pair of vertices from Z_u^\downarrow . On the other hand, all vertices of Z_u^\uparrow are ancestors of u in Q , thus every pair of them is bound by the ancestor-descendant relation in Q .

Claim 5.4.5. *For every k we have $\max_{y \in V(G)} \text{adm}_k[G+F, L, y] \leq 3 \cdot \max_{y \in V(G)} |S_{2k}[G, L, y]|$.*

Proof of claim. Write $H = G + F$. Let $F_{\text{new}} = F \setminus E(G)$ be the set of edges from F that were not already present in G . If an edge $e \in F_{\text{new}}$ belongs also to F_u for some $u \in V(G)$, then we know that u cannot be an endpoint of e . This is because each edge joining a vertex u with one of its children in U is already present in G . We say that the vertex u is the *origin* of an edge $e \in F_{\text{new}} \cap F_u$, and denote it by $a(e)$. Notice that then both endpoints of e are children of $a(e)$ in the tree U , and hence $a(e)$ is adjacent to both the endpoints of e in G .

To give an upper bound on $\max_{y \in V(G)} \text{adm}_k[G+F, L, y]$, let us fix a vertex $u \in V(G)$ and a family of paths \mathcal{P} in H such that

- each path in \mathcal{P} has length at most k , starts in u , ends in a vertex smaller than u in L , and all its internal vertices are larger than u in L ;
- the paths in \mathcal{P} are pairwise vertex-disjoint, apart from the starting vertex u .

For each path $P \in \mathcal{P}$, we define a walk P' in G as follows. For every edge $e = xy$ from F_{new} traversed on P , replace the usage of this edge on P by the following detour of length 2: $x-a(e)-y$. Notice that P' is a walk in the graph G , it starts in u , ends in the same vertex as P , and has length at most $2k$. Next, we define $v(P)$ to be the first vertex on P' (that is, the closest to u on P') that does not belong to G_u . Since the endpoint of P' that is not u

does not belong to G_u , such a vertex exists. Finally, let P'' be the prefix of P' from u to the first visit of $v(P)$ on P' (from the side of u). Observe that the predecessor of $v(P)$ on P'' belongs to G_u and is a neighbour of $v(P)$ in G (since P'' is a walk in G). Hence $v(P)$ has to be an ancestor of u in Q . We find that P'' is a walk of length at most $2k$ in G , it starts in u , ends in $v(P)$, and all its internal vertices belong to G_u , so in particular they are not smaller than u in L . This means that P'' certifies that $v(P) \in S_{2k}[G, L, u]$.

In order to prove the bound on $\max_{y \in V(G)} \text{adm}_k[H, L, y]$, it suffices to prove the following: For each vertex v that is an ancestor of u in Q , there can be at most three paths $P \in \mathcal{P}$ for which $v = v(P)$. To this end, we fix a vertex v that is an ancestor of u in Q and proceed by a case distinction on how a path P with $v = v(P)$ may behave.

Suppose first that v is the endpoint of P other than u , equivalently the endpoint of P' other than u . (For example, $u = 1$, $P = 1, 11, 21, 0$, $P' = 1, 11, 1, 21, 0$ and $v = 0$, in Figures 5.1 and 5.2.) However, the paths of \mathcal{P} are pairwise vertex-disjoint, apart from the starting vertex u , hence there can be at most one path P from \mathcal{P} for which v is an endpoint. Thus, this case contributes at most one path P for which $v = v(P)$.

Next suppose that v is an internal vertex of the walk P' ; in particular, it is not the endpoint of P other than u . (For example, $u = 6$, $P = 6, 11, 21, 0$, $P' = 6, 11, 1, 21, 0$ and $v = 1$, in Figures 5.1 and 5.2.) Since the only vertex traversed by P that is smaller than u in L is this other endpoint of P , and v is smaller than u in L due to being its ancestor in Q , it follows that each visit of v on P' is due to having $v = a(e)$ for some edge $e \in F_{\text{new}}$ traversed on P . Select e to be such an edge corresponding to the first visit of v on P' . Let $e = xy$, where x lies closer to u on P than y . (That is, in our figures, $x = 11$ and $y = 21$.) Since v was chosen as the first vertex on P' that does not belong to G_u , we have $x \in G_u$.

Since $v = a(e) = a(xy)$, either $x \in Z_v^\downarrow$ or $x \in Z_v^\uparrow$. Note that the second possibility cannot happen, because then v would be a descendant of x in Q , hence v would belong to G_u , due to $x \in G_u$; a contradiction. This means $x \in Z_v^\downarrow$.

Recall that, by construction, Z_v^\downarrow contains at most one vertex from each subtree of Q rooted at a child of v . Since v is an ancestor of u in Q , we infer that x has to be the unique vertex of Z_v^\downarrow that belongs to G_u . In the construction of F_v , however, we added only at most two edges of F_v incident to this unique vertex: at most one to its predecessor on the enumeration of the

children of v , and at most one to its successor. Since paths from \mathcal{P} are pairwise vertex-disjoint in H , apart from the starting vertex u , only at most two paths from \mathcal{P} can use any of these two edges (actually, only at most one unless $x = u$). Only for these two paths we can have $v = a(e)$. Thus, this case contributes at most two paths P for which $v = v(P)$, completing the proof of the claim. \lrcorner

We conclude the proof by summarising the construction: first construct the tree U , and then construct the tree T . (Note that we do not need to compute Q : we use it only in the analysis.) By Claims 5.4.4 and 5.4.5, T satisfies the required properties. \square

5.5 An application of Theorem 5.1.11

Theorem 5.1.11, and related results such as 5.1.9, find their original motivation in Model Theory. Here, for the sake of completeness, we sketch the application which motivated these results.

5.5.1 First-order logic

We first need to draw upon some basic notions of first-order logic. In this subsection we follow [23] closely.

A (*finite and purely relational*) *vocabulary* τ is a finite set $\{R_1, \dots, R_s\}$ of *relation symbols*, where each relation symbol R_i has an associated *arity* a_i (that is, R_i relates a_i -tuples). A *structure* \mathfrak{A} of a vocabulary τ (or simply, a τ -*structure*) consists of a set A (the *universe* of \mathfrak{A}) and a relation $R_i(\mathfrak{A}) \subseteq A^{a_i}$ for each relation symbol $R_i \in \tau$. A structure \mathfrak{A} is finite if its universe A is finite. We shall write $Rv_1 \dots v_\ell$ instead of $(v_1, \dots, v_\ell) \in R(\mathfrak{A})$.

Using this terminology, we can define a graph in the following way. We consider the vocabulary $\tau = \{E\}$ formed by a single binary relation symbol E . A (finite, undirected, loopless) graph is a finite τ -structure $\mathfrak{G} = (V, E(\mathfrak{G}))$ such that

- (a) there exists no $v \in V$, such that Evv ;
- (b) for all $u, v \in V$, if Euv , then Evu .

We now proceed to define first-order logic formulae. For a fixed vocabulary τ , every formula will be a string of symbols taken from the alphabet consisting of:

- $v_1, v_2, v_3 \dots$ (a countably infinite set of *variables*)
- \neg, \vee (the *connectives not, or*)
- \exists (the *existential quantifier*)
- $=$ (the *equality symbol*)
- $), ($
- the symbols in τ .

A *term* of a vocabulary is a variable or a constant in τ .

The *formulae* of first-order logic of a vocabulary τ are those strings which are obtained by finitely many applications of the following rules:

- (F1) If t_1 and t_2 are terms, then $t_1 = t_2$ is a formula.
- (F2) If R in τ is r -ary and t_1, \dots, t_r are terms, then Rt_1, \dots, t_r is a formula.
- (F3) If φ is a formula, then $\neg\varphi$ is a formula.
- (F4) If φ and ψ are formulae, then $\varphi \vee \psi$ is a formula.
- (F5) If φ is a formula and x a variable, then $\exists x\varphi$ is a formula.

Denote by $\text{FO}[\tau]$ the set of formulae of first-order logic of vocabulary τ . For two formulae φ and ψ , we use $(\varphi \rightarrow \psi)$ and $(\forall v\varphi)$ as abbreviations for the formulae $(\neg\varphi \vee \psi)$ and $\neg\exists v\neg\varphi$, respectively.

The axioms of graphs stated above have the following formalisations in $\text{FO}[\{E\}]$:

- (a) $\forall v\neg Evv$
- (b) $\forall v\forall u(Evu \rightarrow Euv)$

5.5.2 Model-checking for successor-invariant first-order logic

If a τ -structure \mathfrak{A} *satisfies* a formula φ (if \mathfrak{A} *models* φ), we write $\mathfrak{A} \models \varphi$. The model-checking problem for first-order logic, denoted $\text{MC}(\text{FO})$, is the problem of deciding for a given finite τ -structure \mathfrak{A} and a formula $\varphi \in \text{FO}[\tau]$ whether $\mathfrak{A} \models \varphi$.

If \mathfrak{A} is a finite τ -structure, then the *Gaifman graph* of \mathfrak{A} , denoted $G(\mathfrak{A})$, is the graph on the vertex set A in which two elements $u, v \in A$ are adjacent if and only if $u \neq v$ and u and v appear together in some relation $R_i(\mathfrak{A})$ of \mathfrak{A} . This construction allows us to import concepts from Graph Theory into the study of general relational structures. For instance, we say that a class \mathcal{C} of finite τ -structures has *bounded expansion* if the graph class $G(\mathcal{C}) = \{G(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{C}\}$

has bounded expansion. Similarly, for $k \in \mathbb{N}$, we could write $\text{adm}_k(\mathfrak{A})$ for $\text{adm}_k(G(\mathfrak{A}))$, and so on.

Let us formalise here some ideas introduced in Chapter 1, about parametrised complexity. The model-checking problem $\text{MC}(\mathcal{L})$ for a logic \mathcal{L} is said to be *fixed-parameter tractable* (*FPT* for short) if it can be solved in time $f(|\varphi|) \cdot |\mathfrak{A}|^c$, for some function f and a constant c independent of φ and \mathfrak{A} . We will denote $\text{MC}(\text{FO})$ restricted to a class of finite τ -structures \mathcal{C} as $\text{MC}(\text{FO}, \mathcal{C})$.

Seese [76] proved that first-order model checking is fixed-parameter tractable on any class of graphs of bounded maximum degree. This result was the starting point of a long series of papers establishing tractability results for first-order model-checking on sparse classes of graphs (see [29] for a survey). This line of research culminated in the theorem of Grohe, Kreutzer, and Siebertz [31] stating that for any class \mathcal{C} of graphs that is closed under taking subgraphs, $\text{MC}(\text{FO}, \mathcal{C})$ is FPT if and only if \mathcal{C} is *nowhere dense*. (Nowhere dense graph classes are a family of sparse graph classes which generalises classes with bounded expansion).

A follow-up question is whether tractability on sparse classes of graphs can be achieved for more expressive logics than FO. A classical extension of first-order logic is the *successor-invariant* first-order logic.

Let A be a set. A *successor relation* on A is a binary relation $S \subseteq A \times A$ such that (A, S) is a directed path of length $|A| - 1$. Let τ be a finite relational vocabulary. A formula $\varphi \in \text{FO}[\tau \cup \{S\}]$ is *successor-invariant* if for all τ -structures \mathfrak{A} and for every pair of successor relations S_1, S_2 on A it holds that $(\mathfrak{A}, S_1) \models \varphi$ if and only if $(\mathfrak{A}, S_2) \models \varphi$. We denote the set of all such successor-invariant first-order formulae by $\text{FO}[\tau_{\text{succ}}]$.

Rossman [73] proved that successor-invariant FO is more expressive than plain FO. However, no separation between successor-invariant FO and plain FO is known on sparse classes of graphs, say of bounded expansion. On the other hand, results implying the collapse of these two logics are known only for relatively restricted settings.

With all of this in mind, we can finally state the application which originally motivated Theorem 5.1.11.

Theorem 5.5.1.

Let τ be a finite and purely relational vocabulary and let \mathcal{C} be a class of τ -structures of bounded expansion. Then there exists an algorithm that, given a finite τ -structure $\mathfrak{A} \in \mathcal{C}$ and a formula $\varphi \in \text{FO}[\tau_{\text{succ}}]$, verifies whether $\mathfrak{A} \models \varphi$ in time $f(|\varphi|) \cdot n \cdot \alpha(n)$, where f is a function and n is the size of the universe of \mathfrak{A} .

In the language of parametrised complexity, Theorem 5.5.1 essentially states that the model-checking problem for successor-invariant first-order formulae is fixed-parameter tractable (with almost linear function, since α represents the inverse of the Ackermann function) on classes of finite structures whose underlying Gaifman graphs belong to a fixed class of bounded expansion.

Fixed-parameter tractability of model-checking successor-invariant FO has been shown earlier for planar graphs [26], graphs excluding a fixed minor [25], and graphs excluding a fixed topological minor [24, 53]. As all the above-mentioned classes have bounded expansion, Theorem 5.5.1 generalises all the previously known results in this area.

We will not explain here how Theorem 5.5.1 follows from Theorem 5.1.11. Let us just mention two key ingredients of the proof.

The first is a standard idea used for all earlier results on model-checking for successor-invariant FO. As we would like to check whether $\mathfrak{A} \models \varphi$ for $\varphi \in \text{FO}[\tau_{\text{succ}}]$, we may compute an arbitrary successor relation S on the universe of \mathfrak{A} , and verify whether $(\mathfrak{A}, S) \models \varphi$. Of course, we will try to compute a successor relation S so that adding it to \mathfrak{A} preserves the structural properties as much as possible, so that model-checking on (\mathfrak{A}, S) can be done efficiently. Ideally, if $G(\mathfrak{A})$ contained a Hamiltonian path, we could add a successor relation without introducing any new edges to $G(\mathfrak{A})$. However, in general this might be impossible.

The other helpful ingredient is that we do not actually have to add a successor relation, but it suffices to add some structural information so that a first-order formula can interpret a successor relation. This approach is known as the *interpretation method* [29] and can be used to reduce model-checking for successor-invariant FO to the plain first-order case. In our concrete case, Eickmeyer, Kawarabayashi, and Kreutzer [25] showed that adding a spanning tree of constant maximum degree is enough to be able to interpret some successor relation.

Hence, our first approach was to try to add a Hamiltonian path to every graph in the class, while preserving the sparsity of the class. Since we were unable to do this for a class of graphs with bounded expansion, we made use of the interpretation method. In this way, adding a tree with degree at most 3 (even for a fixed ordering) was enough to reduce model-checking for successor-invariant FO to the plain first-order case. And, as we mentioned earlier, first-order model-checking is known to be fixed parameter tractable even for nowhere dense classes, which generalise classes with bounded expansion.

Chapter 6

Open problems and further directions

In this chapter we collect some open problems which arise from our study of different chromatic and structural properties of sparse graph classes.

6.1 Chromatic number of exact distance graphs

6.1.1 Mind the gap

Whenever distinct upper and lower bounds are given for some combinatorial parameter, the obvious question arises: *where should the bounds meet?*

In Chapters 3 and 4 we give bounds on the chromatic number of exact distance graphs for various types of graph classes. In general, the difference between the best lower and upper bounds is still quite large. But there are some graph classes for which the gap is not so large and for which we could hope to obtain tight bounds, or at least bounds for which the dependency on the distance under consideration has the right order.

When considering odd distances, one trivial example for which there are tight bounds is the class of trees. We noted at the end of Section 3.4, that trees, and in fact all bipartite graphs G satisfy $\chi(G^{\lfloor p \rfloor}) \leq \chi(G^{\lceil p \rceil}) \leq 2$ for every odd p .

Moving from trees to graphs with tree-width at most 2, in Figure 3.1 we construct an outerplanar graph G_4 satisfying $\chi(G_4^{\lfloor 3 \rfloor}) = 5$. In Section 3.3 we show that any graph G with

tree-width at most 2 satisfies $\chi(G^{[\lceil 3 \rceil]}) \leq 13$. Hence, for the class of graphs with tree-width at most 2, the bounds are not far apart when we consider exact distance-3 graphs. The gap is even smaller for chordal graphs of tree-width at most 2, since the graph G_4 happens to be chordal and we show, through Theorem 4.1.4, that any graph G in this class satisfies $\chi(G^{[\lceil 3 \rceil]}) \leq 12$.

What about larger distances? Notice that none of the lower bounds we provide in Chapter 3 increase with the distance considered. Due to the difficulty to provide lower bounds which depend on the distance, the following question, attributed to Van den Heuvel and Naserasr, was asked in [64]: Is there a constant C such that for every odd integer p and every planar graph G we have $\chi(G^{[\lceil p \rceil]}) \leq C$? Very recently, Bousquet, Esperet, Harutyunyan, and de Joannis de Verclos [13] gave a negative answer to this question by constructing a family of chordal outerplanar graphs U_3, U_5, \dots with clique number 3 such that for every odd $p \geq 3$ we have $\chi(U_p^{[\lceil p \rceil]}) \in \Omega(\frac{p}{\log(p)})$. We prove, in Section 3.3 that if G has tree-width at most t then $\chi(G^{[\lceil p \rceil]}) \in \mathcal{O}(p^{t-1})$. This means that graphs G of tree-width at most 2 satisfy $\chi(G^{[\lceil p \rceil]}) \in \mathcal{O}(p)$. Through Theorem 4.1.4, we also prove that there is a linear (in p) upper bound for $\chi(G^{[\lceil p \rceil]})$ when G is chordal. Hence, for graphs of tree-width at most 2, and for chordal graphs, we are close to having the right order in terms of the distance considered.

One interesting result of Chapter 3 is the fact that we prove that $\chi(G^{[\lceil 3 \rceil]}) \leq 143$ if G is planar. Although this upper bound is much better than those previously known, it is still far from the best lower bound known: 7. We hope that Proposition 4.1.7 may help to narrow this gap, as it reduces the problem to the case of maximal planar graphs.

6.1.2 Obtaining upper bounds for more graph classes

In Chapter 3, we obtain two alternative proofs for a result of Nešetřil and Ossona de Mendez [63] which tells us that for every class with bounded expansion \mathcal{K} and every odd p , there is a constant $C = C(\mathcal{K}, p)$ such that, for every graph $G \in \mathcal{K}$, we have $\chi(G^{[\lceil p \rceil]}) \leq C$. However, there are classes of graphs that are not “sparse” and yet allow for the existence of such a function. An example which we have mentioned before is the class of bipartite graphs, for which each constant can be taken to be equal to 2. Hence, one could ask what is the largest type of graph classes which allows for constant bounds on $\chi(G^{[\lceil p \rceil]})$ to exist for every

odd p .

A first approach could be to consider which other classes, sparse or not, allow for such constants to exist. A good candidate seems to be found in classes with bounded *rank-width*. The notion of rank-width, introduced by Oum and Seymour [66], aims to extend that of tree-width by allowing more “dense” graphs classes to have small rank-width. The following theorem follows directly from results of Gurski and Wanke [33] and Oum and Seymour [66].

Theorem 6.1.1 (Gurski and Wanke [33]; Oum and Seymour [66]).

Let \mathcal{K} be a class with bounded rank-width. If \mathcal{K} does not have bounded tree-width, then for every $n > 1$ there is a graph $G_n \in \mathcal{K}$ such that $K_{n,n}$ is a subgraph of G_n .

Hence, a class with bounded rank-width either has bounded-tree width, for which we have constant bounds on $\chi(G^{[p]})$ for all odd p , or has an infinite set of “obstructions” which are bipartite, for which we also have constant bounds on $\chi(G^{[p]})$ for all odd p . Unfortunately, we note that for every odd p there is a family of graphs \mathcal{K}_p with bounded rank-width such that there is no constant upper bound on $\chi(G^{[p]})$ for all $G \in \mathcal{K}_p$. Nevertheless, we could still expect to find upper bounds on $\chi(G^{[p]})$ in terms of $\omega(G^{[p]})$ for p odd.

At the beginning of Chapter 2, we mention that classes with bounded expansion can be characterised through the notion of low tree-width colourings. Recently, Kwon, Pilipczuk, and Siebertz [57] introduced the notion of classes with *low rank-width colourings*, which can be defined just as classes with low tree-width colourings, by substituting the word “tree-width” by the word “rank-width”. Classes with bounded expansion have low rank-width colourings, and so does the class of all complete bipartite graphs. Hence, this type of class generalises all of the classes we know to have constant bounds on $\chi(G^{[p]})$ to exist for every odd p . If classes with bounded rank-width do admit upper bounds on $\chi(G^{[p]})$ in terms of $\omega(G^{[p]})$, for odd p , then one could consider whether classes with low rank-width colourings also admit such upper bounds.

An area that is ripe for further research is the chromatic number of exact distance graphs with even distance. Improving on various results for odd p , Theorem 3.1.8 gives a function f such that $\chi(G^{[p]}) \leq f(\text{wcol}_p(G))$ for every graph G . As argued in Chapter 3, this result is best possible in the sense that we cannot replace $\text{wcol}_p(G)$ with $\text{wcol}_i(G)$, for $i \leq p - 1$. Theorem 3.1.7 (b) gives a first result for even distances, telling us that $\chi(G^{[p]}) \leq \text{wcol}_{2p}(G)$.

$\Delta(G)$. Apart from this, there is very little we know about the dependencies between $\chi(G^{[hp]})$ and the weak colouring numbers.

It is well-known, and easy to prove (see for example [54]), that for every graph G we have $\chi(G^2) \leq (2 \cdot \text{col}(G) - 3) \cdot \Delta(G)$, hence certainly $\chi(G^{[h^2]}) \leq (2 \cdot \text{col}(G) - 3) \cdot \Delta(G)$. This suggests that there might exist a function φ such that $\chi(G^{[hp]}) \leq \varphi(\text{wcol}_{p-1}(G)) \cdot \Delta(G)$, or even $\chi(G^{[hp]}) \leq \varphi(\text{wcol}_{p/2}(G)) \cdot \Delta(G)$. We have not been able to prove such a result. Neither do we know what the best value of $r(p)$ should be such that a result of the form $\chi(G^{[hp]}) \leq \varphi(\text{wcol}_{r(p)}(G)) \cdot \Delta(G)$ is possible for even p .

6.1.3 Considering multiple odd distances

A consequence of Theorem 4.1.5 (a) is that for every odd p , there is a constant $N_{p,t}$ such that $\chi(G^{[h^1]} \cup G^{[h^3]} \cup \dots \cup G^{[hp]}) \leq N_{p,t}$ for all chordal graphs with clique number t . Nešetřil and Ossona de Mendez [64] gave a construction that shows that this upper bound must grow with p , even for outerplanar chordal graphs. Figure 6.1 gives a simpler construction with the same property. However, we note that the construction given by Nešetřil and Ossona de Mendez can be generalised to show that for every $t \geq 3$ and odd positive p , there is a chordal graph G with clique number t such that $\chi(G^{[h^1]} \cup G^{[h^3]} \cup \dots \cup G^{[hp]}) \in \Omega(t^{\lfloor p/2 \rfloor + 1})$. Meanwhile, Theorem 4.1.5 (a) gives that $\chi(G^{[h^1]} \cup G^{[h^3]} \cup \dots \cup G^{[hp]}) \in \mathcal{O}(t^{2\lfloor p/2 \rfloor + 2})$ for these graphs.



Figure 6.1: Outerplanar chordal graphs G for which $\omega(G^{odd})$, and hence $\chi(G^{odd})$, can be arbitrarily large.

For a graph G , a natural generalisation of $G^{[h^1]} \cup G^{[h^3]} \cup \dots \cup G^{[hp]}$ is the graph G^{odd} , which has the same vertex set as G , and xy is an edge in G^{odd} if and only if x and y have odd distance. Both of the constructions mentioned above tell us that even for outerplanar graphs G the chromatic number of G^{odd} can be arbitrarily large, because the clique number $\omega(G^{odd})$ can be arbitrarily large. This inspired the following question of Thomassé, which appeared in [63] (see also [64]).

Problem 6.1.2 ([63, Problem 11.2]).

Is there a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every planar graph G we have $\chi(G^{odd}) \leq f(\omega(G^{odd}))$?

6.2 Generalised colouring numbers

In Chapter 2, we give upper bounds for the strong and the weak colouring numbers of various minor closed classes. Section 2.1 already includes a detailed discussion on the sharpness of these bounds. In general, these upper bounds are very close to having the right order. Nevertheless, the importance of these parameters, and even their relationship to the chromatic number of exact distance graphs, serve as strong motivation for finding improved bounds in the cases where this is possible.

A consequence of Theorem 2.1.2 is that every graph G excluding a fixed minor H satisfies $\text{adm}_k(G) \in \mathcal{O}(k)$. The fact that there is a linear upper bound on the k -admissibility was already known, even for classes excluding a fixed topological minor [30]. Nevertheless, it would be interesting to see whether $\text{adm}_k(G)$ can have sublinear upper bounds for minor closed classes (other than classes with bounded tree-width, where there is a uniform bound).

Proposition 2.1.1 tells us that for every graph G we have $\text{scol}_\infty(G) = \text{tw}(G) + 1$ and $\text{wcol}_\infty(G) = \text{td}(G)$. It would be interesting to find some structural parameter which equals $\text{adm}_\infty(G)$.

One of our main tools for finding upper bounds on the weak colouring numbers is Lemma 2.3.3. This lemma tells us that by contracting the subgraphs of a connected decomposition of width t , we obtain a graph with tree-width at most t . By proving, through Lemma 2.4.1, that classes excluding a fixed complete minor admit decompositions of small width, we obtain an interesting structural result for these classes. In Chapter 3, we make use of these ideas in order to obtain upper bounds for the distance- k -colouring numbers. Moreover, they are used in [40] to obtain upper bounds on various improper colourings related to Hadwiger's Conjecture. It would be interesting to see where else these promising structural results can be put to good use.

6.3 Adding graph classes

In Chapter 5, we introduce a notion of addition of graph classes. We focus on obtaining results which guarantee that two sparse classes can be summed so as to obtain another sparse class.

One of our results tells us that a class with bounded tree-width and the class of all paths can be summed so as to obtain a class with bounded tree-width. A consequence of Theorem 5.1.5 is that a proper minor closed class and the class of all paths can be summed so as to obtain a class with bounded expansion. Nevertheless, Theorem 5.1.5 is far more general as it considers the addition of any two minor closed classes, and its proof uses very strong results. Hence, it might be possible to prove that a minor closed class and the class of all paths can be summed so as to obtain a (proper) minor closed class, or at least a class which excludes a fixed topological minor.

In contrast with the very general results that we can prove when adding two minor closed classes, we do not know whether a class excluding a fixed topological minor (or, indeed, any class with bounded expansion) and the class of all paths can be summed so as to obtain a class with bounded expansion. In fact, getting a partial result in this direction is one of the main results of Chapter 5 and the proof is far from trivial. We note, however, that any class with bounded expansion and the class of all paths can be summed to a class with low rank-width colourings. This is a consequence of the following observations:

- It is well known that the cube G^3 of a connected graph G contains a Hamiltonian path (in fact, every two vertices are connected by a Hamiltonian path [41, 77]).
- Theorem 2 of [57] tells us that for every class \mathcal{C} with bounded expansion, the class $\{G^3 \mid G \in \mathcal{C}\}$, of cubes of graphs in \mathcal{C} , is a class with low rank-width colourings.

One of the very strong results which allows us to prove Theorem 5.1.5, is a theorem of Blankenship [10] which tells us that minor closed classes have bounded page number. The proof of this theorem, as it appears in [10], is very long and uses the structural theory of Robertson and Seymour. It would be useful to have a simpler proof of this result. Perhaps our results on connected decompositions of small width for minor closed classes could help in this regard.

Finally, it would be interesting to see some negative results for the question: when are

sparse graph classes closed under addition? It would be very nice to find examples of relatively simple graph classes that cannot be summed so as to obtain a (proper) minor closed class, a class excluding a fixed topological minor, or even a class with bounded expansion.

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